THE 1996 ASIAN PACIFIC MATHEMATICAL OLYMPIAD

Time allowed: 4 hours
NO calculators are to be used.
Each question is worth seven points.

Question 1

Let $ABCD$ be a quadrilateral $AB = BC = CD = DA$. Let MN and PQ be two segments perpendicular to the diagonal BD and such that the distance between them is $d > BD/2$, with $M \in AD$, $N \in DC$, $P \in AB$, and $Q \in BC$. Show that the perimeter of hexagon $AMNCQP$ does not depend on the position of MN and PQ so long as the distance between them remains constant.

Question 2

Let m and n be positive integers such that $n \leq m$. Prove that

$$2^n n! \leq \frac{(m+n)!}{(m-n)!} \leq (m^2+m)^n \ .$$

Question 3

Let P_1, P_2, P_3, P_4 be four points on a circle, and let I_1 be the incentre of the triangle $P_2P_3P_4$; I_2 be the incentre of the triangle $P_1P_3P_4$; I_3 be the incentre of the triangle $P_1P_2P_4$; I_4 be the incentre of the triangle $P_1P_2P_3$. Prove that I_1, I_2, I_3, I_4 are the vertices of a rectangle.

Question 4

The National Marriage Council wishes to invite n couples to form 17 discussion groups under the following conditions:

1. All members of a group must be of the same sex; i.e. they are either all male or all female.
2. The difference in the size of any two groups is 0 or 1.
3. All groups have at least 1 member.
4. Each person must belong to one and only one group.

Find all values of n, $n \leq 1996$, for which this is possible. Justify your answer.

Question 5

Let a, b, c be the lengths of the sides of a triangle. Prove that

$$\sqrt{a+b-c} + \sqrt{b+c-a} + \sqrt{c+a-b} \leq \sqrt{a} + \sqrt{b} + \sqrt{c} \ ,$$

and determine when equality occurs.