2011 APMO PROBLEMS

<u>*Time allowed: 4 hours*</u> <u>*Each problem is worth 7 points*</u>

**The contest problems are to be kept confidential until they are posted on the official APMO website (http://www.mmjp.or.jp/competitions/APMO). Please do not disclose nor discuss the problems over the internet until that date. Calculators are not allowed to use.*

Problem 1. Let a, b, c be positive integers. Prove that it is impossible to have all of the three numbers $a^2 + b + c,\ b^2 + c + a,\ c^2 + a + b$ to be perfect squares.

Problem 2. Five points A_1, A_2, A_3, A_4, A_5 lie on a plane in such a way that no three among them lie on a same straight line. Determine the maximum possible value that the minimum value for the angles $\angle A_i A_j A_k$ can take where i, j, k are distinct integers between 1 and 5.

Problem 3. Let ABC be an acute triangle with $\angle BAC = 30°$. The internal and external angle bisectors of $\angle ABC$ meet the line AC at B_1 and B_2, respectively, and the internal and external angle bisectors of $\angle ACB$ meet the line AB at C_1 and C_2, respectively. Suppose that the circles with diameters B_1B_2 and C_1C_2 meet inside the triangle ABC at point P. Prove that $\angle BPC = 90°$.

Problem 4. Let n be a fixed positive odd integer. Take $m + 2$ **distinct** points $P_0, P_1, \cdots, P_{m+1}$ (where m is a non-negative integer) on the coordinate plane in such a way that the following 3 conditions are satisfied:
(1) $P_0 = (0, 1),\ P_{m+1} = (n+1, n)$, and for each integer i, $1 \le i \le m$, both x- and y- coordinates of P_i are integers lying in between 1 and n (1 and n inclusive).
(2) For each integer i, $0 \le i \le m$, P_iP_{i+1} is parallel to the x-axis if i is even, and is parallel to the y-axis if i is odd.
(3) For each pair i, j with $0 \le i < j \le m$, line segments P_iP_{i+1} and P_jP_{j+1} share at most 1 point.

Determine the maximum possible value that m can take.

Problem 5. Determine all functions $f : \mathbf{R} \to \mathbf{R}$, where $\mathbf{R}$ is the set of all real numbers, satisfying the following 2 conditions:
(1) There exists a real number M such that for every real number x, $f(x) < M$ is satisfied.
(2) For every pair of real numbers x and y,

$$f(xf(y)) + yf(x) = xf(y) + f(xy)$$

is satisfied.