Canadian Mathematical Olympiad
1990

Problem 1

A competition involving $n \geq 2$ players was held over k days. On each day, the players received scores of $1, 2, 3, \ldots, n$ points with no two players receiving the same score. At the end of the k days, it was found that each player had exactly 26 points in total. Determine all pairs (n, k) for which this is possible.

Problem 2

A set of $\frac{1}{2}n(n+1)$ distinct numbers is arranged at random in a triangular array:

$$\begin{array}{ccccccccc} & & & & \times & & & & \\ & & & \times & & \times & & & \\ & & \times & & \times & & \times & & \\ & & \vdots & & \vdots & & \vdots & & \\ \times & \times & \cdot & & \cdot & & \cdot & \times & \times \end{array}$$

Let M_k be the largest number in the k-th row from the top. Find the probability that

$$M_1 < M_2 < M_3 < \cdots < M_n.$$

Problem 3

Let $ABCD$ be a convex quadrilateral inscribed in a circle, and let diagonals AC and BD meet at X. The perpendiculars from X meet the sides AB, BC, CD, DA at A', B', C', D' respectively. Prove that

$$|A'B'| + |C'D'| = |A'D'| + |B'C'|.$$

($|A'B'|$ is the length of line segment $A'B'$, *etc.*)

Problem 4

A particle can travel at speeds up to 2 metres per second along the x-axis, and up to 1 metre per second elsewhere in the plane. Provide a labelled sketch of the region which can be reached within one second by the particle starting at the origin.

Problem 5

Suppose that a function f defined on the positive integers satisfies

$$f(1) = 1, \quad f(2) = 2,$$
$$f(n+2) = f\big(n + 2 - f(n+1)\big) + f\big(n + 1 - f(n)\big) \quad (n \geq 1).$$

(a) Show that
 (i) $0 \leq f(n+1) - f(n) \leq 1$
 (ii) if $f(n)$ is odd, then $f(n+1) = f(n) + 1$.

(b) Determine, with justification, all values of n for which

$$f(n) = 2^{10} + 1.$$