Canadian Mathematical Olympiad
1996

Problem 1

If α, β, γ are the roots of $x^3 - x - 1 = 0$, compute

$$\frac{1+\alpha}{1-\alpha} + \frac{1+\beta}{1-\beta} + \frac{1+\gamma}{1-\gamma}.$$

Problem 2

Find all real solutions to the following system of equations. Carefully justify your answer.

$$\begin{cases} \dfrac{4x^2}{1+4x^2} = y \\ \dfrac{4y^2}{1+4y^2} = z \\ \dfrac{4z^2}{1+4z^2} = x \end{cases}$$

Problem 3

We denote an arbitrary permutation of the integers $1, \ldots, n$ by $a_1, \ldots, a_n$. Let $f(n)$ be the number of these permutations such that

(i) $a_1 = 1$;
(ii) $|a_i - a_{i+1}| \leq 2, \quad i = 1, \ldots, n-1$.

Determine whether $f(1996)$ is divisible by 3.

Problem 4

Let $\triangle ABC$ be an isosceles triangle with $AB = AC$. Suppose that the angle bisector of $\angle B$ meets AC at D and that $BC = BD + AD$. Determine $\angle A$.

Problem 5

Let $r_1, r_2, \ldots, r_m$ be a given set of m positive rational numbers such that $\sum_{k=1}^{m} r_k = 1$. Define the function f by $f(n) = n - \sum_{k=1}^{m}[r_k n]$ for each positive integer n. Determine the minimum and maximum values of $f(n)$. Here $[x]$ denotes the greatest integer less than or equal to x