THE 2000 CANADIAN MATHEMATICAL OLYMPIAD

1. At 12:00 noon, Anne, Beth and Carmen begin running laps around a circular track of length three hundred meters, all starting from the same point on the track. Each jogger maintains a constant speed in one of the two possible directions for an indefinite period of time. Show that if Anne's speed is different from the other two speeds, then at some later time Anne will be at least one hundred meters from each of the other runners. (Here, distance is measured along the shorter of the two arcs separating two runners.)

2. A *permutation* of the integers $1901, 1902, \ldots, 2000$ is a sequence $a_1, a_2, \ldots, a_{100}$ in which each of those integers appears exactly once. Given such a permutation, we form the sequence of partial sums

$$s_1 = a_1, \quad s_2 = a_1 + a_2, \quad s_3 = a_1 + a_2 + a_3, \quad \ldots \quad, \quad s_{100} = a_1 + a_2 + \cdots + a_{100}.$$

How many of these permutations will have no terms of the sequence $s_1, \ldots, s_{100}$ divisible by three?

3. Let $A = (a_1, a_2, \ldots, a_{2000})$ be a sequence of integers each lying in the interval $[-1000, 1000]$. Suppose that the entries in A sum to 1. Show that some nonempty subsequence of A sums to zero.

4. Let $ABCD$ be a convex quadrilateral with

$$\angle CBD = 2\angle ADB,$$
$$\angle ABD = 2\angle CDB$$
$$\text{and} \qquad AB = CB.$$

Prove that $AD = CD$.

5. Suppose that the real numbers $a_1, a_2, \ldots, a_{100}$ satisfy

$$a_1 \geq a_2 \geq \cdots \geq a_{100} \geq 0,$$
$$a_1 + a_2 \leq 100$$
$$\text{and} \qquad a_3 + a_4 + \cdots + a_{100} \leq 100.$$

Determine the maximum possible value of $a_1^2 + a_2^2 + \cdots + a_{100}^2$, and find all possible sequences $a_1, a_2, \ldots, a_{100}$ which achieve this maximum.