CMO 1996
SOLUTIONS

QUESTION 1

Solution .

If $f(x) = x^3 - x - 1 = (x-\alpha)(x-\beta)(x-\gamma)$ has roots α, β, γ standard results about roots of polynomials give $\alpha+\beta+\gamma = 0$, $\alpha\beta+\alpha\gamma+\beta\gamma = -1$, and $\alpha\beta\gamma = 1$.

Then

$$S = \frac{1+\alpha}{1-\alpha} + \frac{1+\beta}{1-\beta} + \frac{1+\gamma}{1-\gamma} = \frac{N}{(1-\alpha)(1-\beta)(1-\gamma)}$$

where the numerator simplifies to

$$\begin{aligned} N &= 3 - (\alpha+\beta+\gamma) - (\alpha\beta+\alpha\gamma+\beta\gamma) + 3\alpha\beta\gamma \\ &= 3 - (0) - (-1) + 3(1) \\ &= 7. \end{aligned}$$

The denominator is $f(1) = -1$ so the required sum is -7.

QUESTION 2

Solution 1.

For any t, $0 \le 4t^2 < 1 + 4t^2$, so $0 \le \dfrac{4t^2}{1+4t^2} < 1$. Thus x, y and z must be non-negative and less than 1.

Observe that if one of x y or z is 0, then $x = y = z = 0$.

If two of the variables are equal, say $x = y$, then the first equation becomes

$$\frac{4x^2}{1+4x^2} = x.$$

This has the solution $x = 0$, which gives $x = y = z = 0$ and $x = \dfrac{1}{2}$ which gives $x = y = z = \dfrac{1}{2}$.

Finally, assume that x, y and z are non-zero and distinct. Without loss of generality we may assume that either $0 < x < y < z < 1$ or $0 < x < z < y < 1$. The two proofs are similar, so we do only the first case.

We will need the fact that $f(t) = \dfrac{4t^2}{1+4t^2}$ is increasing on the interval $(0, 1)$.

To prove this, if $0 < s < t < 1$ then

$$\begin{aligned} f(t) - f(s) &= \frac{4t^2}{1+4t^2} - \frac{4s^2}{1+4s^2} \\ &= \frac{4t^2 - 4s^2}{(1+4s^2)(1+4t^2)} \\ &> 0. \end{aligned}$$

So $0 < x < y < z \Rightarrow f(x) = y < f(y) = z < f(z) = x$, a contradiction.

Hence $x = y = z = 0$ and $x = y = z = \dfrac{1}{2}$ are the only real solutions.

Solution 2.

Notice that x, y and z are non-negative. Adding the three equations gives

$$x + y + z = \frac{4z^2}{1+4z^2} + \frac{4x^2}{1+4x^2} + \frac{4y^2}{1+4y^2}.$$

This can be rearranged to give

$$\frac{x(2x-1)^2}{1+4x^2} + \frac{y(2y-1)^2}{1+4y^2} + \frac{z(2z-1)^2}{1+4z^2} = 0.$$

Since each term is non-negative, each term must be 0, and hence each variable is either 0 or $\dfrac{1}{2}$. The original equations then show that $x = y = z = 0$ and $x = y = z = \dfrac{1}{2}$ are the only two solutions.

Solution 3.

Notice that x, y, and z are non-negative. Multiply both sides of the inequality

$$\frac{y}{1+4y^2} \geq 0$$

by $(2y-1)^2$, and rearrange to obtain

$$y - \frac{4y^2}{1+4y^2} \geq 0,$$

and hence that $y \geq z$. Similarly, $z \geq x$, and $x \geq y$. Hence, $x = y = z$ and, as in Solution 1, the two solutions follow.

Solution 4.

As for solution 1, note that $x = y = z = 0$ is a solution and any other solution will have each of x, y and z positive.

The arithmetic-geometric mean inequality (or direct computation) shows that $\dfrac{1+4x^2}{2} \geq \sqrt{1 \cdot 4x^2} = 2x$ and hence $x \geq \dfrac{4x^2}{1+4x^2} = y$, with equality if and only if $1 = 4x^2$ – that is, $x = \dfrac{1}{2}$. Similarly, $y \geq z$ with equality if and only if $y = \dfrac{1}{2}$ and $z \geq x$ with equality if and only if $z = \dfrac{1}{2}$. Adding $x \geq y$, $y \geq z$ and $z \geq x$ gives $x+y+x \geq x+y+z$. Thus equality must occur in each inequality, so $x = y = z = \dfrac{1}{2}$.

QUESTION 3

Solution.

Let $a_1, a_2, \ldots, a_n$ be a permutation of $1, 2, \ldots, n$ with properties (i) and (ii).

A crucial observation, needed in Case II (b) is the following: If a_k and a_{k+1} are consecutive integers (i.e. $a_{k+1} = a_k \pm 1$), then the terms to the right of a_{k+1} (also to the left of a_k) are either all less than both a_k and a_{k+1} or all greater than both a_k and a_{k+1}.

Since $a_1 = 1$, by (ii) a_2 is either 2 or 3.

CASE I: Suppose $a_2 = 2$. Then $a_3, a_4, \ldots, a_n$ is a permutation of $3, 4, \ldots, n$. Thus $a_2, a_3, \ldots, a_n$ is a permutation of $2, 3, \ldots, n$ with $a_2 = 2$ and property (ii). Clearly there are $f(n-1)$ such permutations.

CASE II: Suppose $a_2 = 3$.

(a) Suppose $a_3 = 2$. Then $a_4, a_5, \ldots, a_n$ is a permutation of $4, 5, \ldots, n$ with $a_4 = 4$ and property (ii). There are $f(n-3)$ such permutations.

(b) Suppose $a_3 \geq 4$. If a_{k+1} is the first even number in the permutation then, because of (ii), $a_1, a_2, \ldots, a_k$ must be $1, 3, 5, \ldots, 2k-1$ (in that order). Then a_{k+1} is either $2k$ or $2k-2$, so that a_k and a_{k+1} are consecutive integers. Applying the crucial observation made above, we deduce that $a_{k+2}, \ldots, a_n$ are all either greater than or smaller than a_k and a_{k+1}. But 2 must be to the right of a_{k+1}. Hence $a_{k+2}, \ldots, a_n$ are the even integers less than a_{k+1}. The only possibility then, is

$$1, 3, 5, \ldots, a_{k-1}, a_k, \ldots, 6, 4, 2.$$

Cases I and II show that

$$f(n) = f(n-1) + f(n-3) + 1, \quad n \geq 4. \qquad (*)$$

Calculating the first few values of $f(n)$ directly gives

$$f(1) = 1,\ f(2) = 1,\ f(3) = 2,\ f(4) = 4,\ f(5) = 6.$$

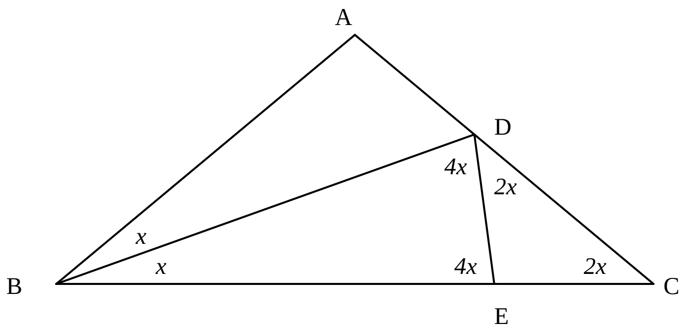
Calculating a few more $f(n)$'s using (*) and mod 3 arithmetic, $f(1) = 1$, $f(2) = 1$, $f(3) = 2$, $f(4) = 1$, $f(5) = 0$, $f(6) = 0$, $f(7) = 2$, $f(8) = 0$, $f(9) = 1$, $f(10) = 1$, $f(11) = 2$. Since $f(1) = f(9)$, $f(2) = f(10)$ and $f(3) = f(11)$ mod 3, (*) shows that $f(a) = f(a \mathit{ mod } 8)$, mod 3, $a \geq 1$.

Hence $f(1996) \equiv f(4) \equiv 1 \pmod 3$ so 3 does not divide $f(1996)$.

QUESTION 4

Solution 1.

Let $BE = BD$ with E on BC, so that $AD = EC$:



By a standard theorem, $\dfrac{AB}{CB} = \dfrac{AD}{DC}$; so in

$\triangle\, CED$ and $\triangle\, CAB$ we have a common angle and

$$\frac{CE}{CD} = \frac{AD}{CD} = \frac{AB}{CB} = \frac{CA}{CB} \, .$$

Thus $\triangle\, CED \sim \triangle\, CAB$, so that $\angle\, CDE = \angle\, DCE = \angle\, ABC = 2x$.
Hence $\angle BDE = \angle BED = 4x$, whence $9x = 180^\circ$ so $x = 20^\circ$.
Thus $\angle A = 180^\circ - 4x = 100^\circ$.

Solution 2.

Apply the law of sines to $\triangle ABD$ and $\triangle BDC$ to get

$$\frac{AD}{BD} = \frac{\sin x}{\sin 4x} \quad \text{and} \quad 1 + \frac{AD}{BD} = \frac{BC}{BD} = \frac{\sin 3x}{\sin 2x} \, .$$

Now massage the resulting trigonometric equation with standard identities to get

$$\sin 2x\,(\sin 4x + \sin x) = \sin 2x\,(\sin 5x + \sin x) \, .$$

Since $0 < 2x < 90^\circ$, we get

$$5x - 90^\circ = 90^\circ - 4x \, ,$$

so that $\angle A = 100^\circ$.

QUESTION 5

Solution.

Let

$$f(n) = n - \sum_{k=1}^{m}[r_k n]$$

$$= n\sum_{k=1}^{m} r_k - \sum_{k=1}^{m}[r_k n]$$

$$= \sum_{k=1}^{m}\{r_k n - [r_k n]\}.$$

Now $0 \leq x - [x] < 1$, and if c is an integer, $(c+x) - [c+x] = x - [x]$.

Hence $0 \leq f(n) < \sum_{k=1}^{m} 1 = m$. Because $f(n)$ is an integer, $0 \leq f(n) \leq m-1$.

To show that $f(n)$ can achieve these bounds for $n > 0$, we assume that $r_k = \dfrac{a_k}{b_k}$ where a_k, b_k are integers; $a_k < b_k$.

Then, if $n = b_1 b_2 \dots b_m,\ (r_k n) - [r_k n] = 0,\ k = 1, 2, \dots, m$ and thus $f(n) = 0$.

Letting $n = b_1 b_2 \dots b_n - 1$, then

$$\begin{aligned} r_k n &= r_k(b_1 b_2 \dots b_m - 1) \\ &= r_k\{(b_1 b_2 \dots b_m - b_k) + b_k - 1)\} \\ &= \text{integer} + r_k(b_k - 1). \end{aligned}$$

This gives

$$r_k n - [r_k n] = r_k(b_k - 1) - [r_k(b_k - 1)]$$

$$= \frac{a_k}{b_k}(b_k - 1) - \left[\frac{a_k}{b_k}(b_k - 1)\right]$$

$$= \left(a_k - \frac{a_k}{b_k}\right) - \left[a_k - \frac{a_k}{b_k}\right]$$

$$= \left(a_k - \frac{a_k}{b_k}\right) - (a_k - 1)$$

$$= 1 - \frac{a_k}{b_k} = 1 - r_k.$$

Hence

$$f(n) = \textstyle\sum_{k=1}^{m}(1 - r_k) = m - 1.$$