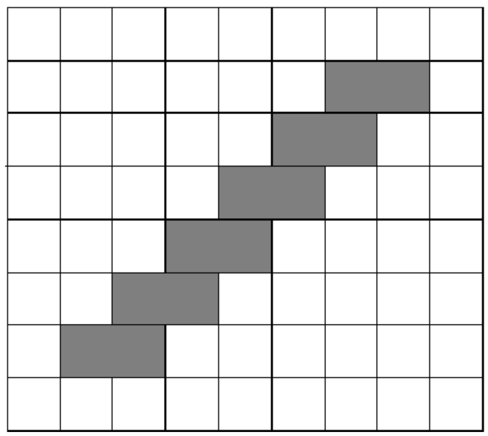
39th Canadian Mathematical Olympiad

Wednesday, March 28, 2007

Solutions to the 2007 CMO paper



Solution to 1. Identify five subsets A, B, C, D, E of the board, where C consists of the squares occupied by the six dominos already placed, B is the upper right corner, D is the lower left corner, A consists of the squares above and to the left of those in $B \cup C \cup D$ and E consists of the squares below and to the right of those in $B \cup C \cup D$. The board can be coloured checkerboard fashion so that A has 13 black and 16 white squares, B a single white square, E 16 black and 13 white squares and D a single black square. Each domino beyond the original six must lie either entirely in $A \cup B \cup D$ or $C \cup B \cup D$, either of which contains at most 14 dominos. Thus, altogether, we cannot have more that $2 \times 14 + 6 = 34$ dominos. This is achievable, by placing 14 dominos in $A \cup D$ and 14 in $E \cup B$.

Solution to 2. If the triangles are isosceles, then they must be congruent and the desired ratio is 1. For, if they share equal side lengths, at least one of these side lengths on one triangle corresponds to the same length on the other. And if they share unequal side lengths, then either equal sides correspond or unequal sides correspond in both directions and the ratio is 1. This falls within the bounds.

Let the triangles be scalene. It is not possible for the same length to be an extreme length (largest or smallest) of both triangles. Therefore, we must have a situation in which the corresponding side lengths of the two triangles are (x, y, z) and (y, z, u) with $x < y < z$ and $y < z < u$. We are given that $y/x = z/y = u/z = r > 1$. Thus, $y = rx$ and $z = ry = r^2x$. From the triangle inequality $z < x + y$, we have that $r^2 < 1 + r$. Since $r^2 - r - 1 < 0$ and $r > 1$, $1 < r < \frac{1}{2}(\sqrt{5} + 1)$. The ratio of the dimensions from the smaller to the larger triangle is $1/r$ which satisfies $\frac{1}{2}(\sqrt{5} - 1) < 1/r < 1$. The result follows.

Solution to 3. (a) Let $f(x) = x^2 + 4$. Then

$$\begin{aligned} f(xy) + f(y - x) - f(y + x) &= (x^2y^2 + 4) + (y - x)^2 + 4 - (y + x)^2 - 4 \\ &= (xy)^2 - 4xy + 4 = (xy - 2)^2 \geq 0 \ . \end{aligned} \tag{1}$$

Thus, $f(x) = x^2 + 4$ satisfies the condition.

(b) Consider (x, y) for which $xy = x + y$. Rewriting this as $(x - 1)(y - 1) = 1$, we find that this has the general solution $(x, y) = (1 + t^{-1}, 1 + t)$, for $t \neq 0$. Plugging this into the inequality, we get that $f(t - t^{-1}) \geq 0$ for all $t \neq 0$. For arbitrary real

u, the equation $t - t^{-1} = u$ leads to the quadratic $t^2 - ut - 1 = 0$ which has a positive discriminant and so a real solution. Hence $f(u) \geq 0$ for each real u.

Comment. The substitution $v = y - x$, $u = y + x$ whose inverse is $x = \frac{1}{2}(u - v)$, $y = \frac{1}{2}(u + v)$ renders the condition as $f(\frac{1}{4}(u^2 - v^2)) + f(v) \geq f(u)$. The same strategy as in the foregoing solution leads to the choice $u = 2 + \sqrt{v^2 + 4}$ and $f(v) \geq 0$ for all v.

Solution to 4 (b). It is straightforward to verify that $a * 1 = 1$ for $a \neq 1$, so that once 1 is included in the list, it can never by removed and so the list terminates with the single value 1.

Solution to 4 (a). There are several ways of approaching (a). It is important to verify that the set $\{x : 0 < x < 1\}$ is closed under the operation $*$ so that it is always defined.

If $0 < a, b < 1$, then

$$0 < \frac{a + b - 2ab}{1 - ab} < 1 \ .$$

The left inequality follows from

$$a + b - 2ab = a(1 - b) + b(1 - a) > 0$$

and the right from

$$1 - \frac{a + b - 2ab}{1 - ab} = \frac{(1 - a)(1 - b)}{1 - ab} > 0 \ .$$

Hence, it will never happen that a set of numbers will contain a pair of reciprocals, and the operation can always be performed.

Solution 1. It can be shown by induction that any two numbers in any of the sets arise from disjoint subsets of S.

Use an induction argument on the number of entries that one starts with. At each stage the number of entries is reduced by one. If we start with n numbers, the final result is

$$\frac{\sigma_1 - 2\sigma_2 + 3\sigma_3 - \cdots + (-1)^{n-1} n\sigma_n}{1 - \sigma_2 + 2\sigma_3 - 3\sigma_4 + \cdots + (-1)^{n-1}(n - 1)\sigma_n} \ ,$$

where σ_i is the symmetric sum of all $\binom{n}{i}$ $i-$fold products of the n elements x_i in the list.

Solution 2. Define

$$a * b = \frac{a + b - 2ab}{1 - ab} \ .$$

This operation is commutative and also associative:

$$a * (b * c) = (a * b) * c = \frac{a + b + c - 2(ab + bc + ca) + 3abc}{1 - (ab + bc + ca) + 2abc} \ .$$

Since the final result amounts to a $*-$product of elements of S with some arrangement of brackets, the result follows.

Solution 3. Let $\phi(x) = x/(1 - x)$ for $0 < x < 1$. This is a one-one function from the open interval $(0, 1)$ to the half line $(0, \infty)$. For any numbers $a, b \in S$, we have that

$$\begin{aligned} \phi\left(\frac{a + b - 2ab}{1 - ab}\right) &= \tfrac{a+b-2ab}{(1-ab)-(a+b-2ab)} = \tfrac{a+b-2ab}{1-a-b+ab} \\ &= \tfrac{a}{1-a} + \tfrac{b}{1-b} = \phi(a) + \phi(b) \ . \end{aligned} \tag{2}$$

Let $T = \{\phi(s) : s \in S\}$. Then replacing a, b in S as indicated corresponds to replacing $\phi(a)$ and $\phi(b)$ in T by $\phi(a) + \phi(b)$ to get a new pair of sets related by ϕ. The final result is the inverse under ϕ of $\sum\{\phi(s) : s \in S\}$.

Solution 4. Let $f(x) = (1 - x)^{-1}$ be defined for positive x unequal to 1. Then $f(x) > 1$ if and only if $0 < x < 1$. Observe that

$$f(x * y) = \frac{1 - xy}{1 - x - y + xy} = \frac{1}{1 - x} + \frac{1}{1 - y} - 1 \ .$$

If $f(x) > 1$ and $f(y) > 1$, then also $f(x * y) > 1$. It follows that if x and y lie in the open interval $(0, 1)$, so does $x * y$. We also note that $f(x)$ is a one-one function.

To each list L, we associate the function $g(L)$ defined by

$$g(L) = \sum \{f(x) : x \in L\} .$$

Let L_n be the given list, and let the subsequent lists be $L_{n-1}, L_{n-2}, \cdots, L_1$, where L_i has i elements. Since $f(x * y) = f(x) + f(y) - 1$, $g(L_i) = g(L_n) - (n - i)$ regardless of the choice that creates each list from its predecessors. Hence $g(L_1) = g(L_n) - (n - 1)$ is fixed. However, $g(L_1) = f(a)$ for some number a with $0 < a < 1$. Hence $a = f^{-1}(g(L_n) - (n - 1))$ is fixed.

Solution to 5 (a). Let I be the incentre of triangle ABC. Since the quadrilateral $AEIF$ has right angles at E and F, it is concyclic, so that Γ_1 passes through I. Similarly, Γ_2 and Γ_3 pass through I, and (a) follows.

Solution to 5 (b). Let ω and I denote the incircle and incentre of triangle ABC, respectively. Observe that, since AI bisects the angle FAE and $AF = AE$, then AI right bisects the segment FE. Similarly, BI right bisects DF and CI right bisects DE.

We invert the diagram through ω. Under this inversion, let the image of A be A', *etc.* Note that the centre I of inversion is collinear with any point and its image under the inversion. Under this inversion, the image of Γ_1 is EF, which makes A' the midpoint of EF. Similarly, B' is the midpoint of DF and C' is the midpoint of DE. Hence, Γ', the image of Γ under this inversion, is the circumcircle of triangle $A'B'C'$, which implies that Γ' is the nine-point circle of triangle DEF.

Since P is the intersection of Γ and Γ_1 other than A, P' is the intersection of Γ' and EF other than A', which means that P' is the foot of the altitude from D to EF. Similarly, Q' is the foot of the altitude from E to DF and R' is the foot of the altitude from F to DE.

Now, let X, Y and Z be the midpoints of arcs BC, AC and AB on Γ respectively. We claim that X lies on PD.

Let X' be the image of X under the inversion, so I, X and X' are collinear. But X is the midpoint of arc BC, so A, A', I, X' and X are collinear. The image of line PD is the circumcircle of triangle $P'ID$, so to prove that X lies on PD, it suffices to prove that points P', I, X' and D are concyclic.

We know that B' is the midpoint of DF, C' is the midpoint of DE and P' is the foot of the altitude from D to EF. Hence, D is the reflection of P' in $B'C'$.

Since $IA' \perp EF$, $IB' \perp DF$ and $IC' \perp DE$, I is the orthocentre of triangle $A'B'C'$. So, X' is the intersection of the altitude from A' to $B'C'$ with the circumcircle of triangle $A'B'C'$. From a wellknown fact, X' is the reflection of I in $B'C'$. This means that $B'C'$ is the perpendicular bisector of both $P'D$ and IX', so that the points P', I, X' and D are concyclic.

Hence, X lies on PD. Similarly, Y lies on QE and Z lies on RF. Thus, to prove that PD, QE and RF are concurrent, it suffices to prove that DX, EY and FZ are concurrent.

To show this, consider tangents to Γ at X, Y and Z. These are parallel to BC, AC and AB, respectively. Hence, the triangle Δ that these tangents define is homothetic to the triangle ABC. Let S be the centre of homothety. Then the homothety taking triangle ABC to Δ takes ω to Γ, and so takes D to X, E to Y and F to Z. Hence DX, EY and FZ concur at S.

Comment. The solution uses the following result: *Suppose ABC is a triangle with orthocentre H and that AH intersects BC at P and the circumcircle of ABC at D. Then $HP = PD$.* The proof is straightforward: Let BH meet AC at Q. Note that $AD \perp BC$ and $BQ \perp AC$. Since $\angle ACB = \angle ADB$,

$$\angle HBC = \angle QBC = 90^\circ - \angle QCB = 90^\circ - \angle ACB = 90^\circ - \angle ADB = \angle DBP ,$$

from which follows the congruence of triangle HBP and DBP and equality of HP and PD.

Solution 2. (a) Let Γ_2 and Γ_3 intersect at J. Then $BDJF$ and $CDJE$ are concyclic. We have that

$$\begin{aligned} \angle FJE &= 360^\circ - (\angle DJF + \angle DJE) \\ &= 360^\circ - (180^\circ - \angle ABC + 180^\circ - \angle ACB) \\ &= \angle ABC + \angle ACB = 180^\circ - \angle FAE . \end{aligned} \tag{3}$$

Hence $AFJE$ is concyclic and so the circumcircles of AEF, BDF and CED pass through J.

(b) [Y. Li] Join RE, RD, RA and RB. In Γ_3, $\angle ERD = \angle ECD = \angle ACB$ and $\angle REC = \angle RDC$. In Γ, $\angle ARB = \angle ACB$. Hence, $\angle ERD = \angle ARB \Longrightarrow \angle ARE = \angle BRD$. Also,

$$\angle AER = 180^\circ - \angle REC = 180^\circ - \angle RDC = \angle BDR .$$

Therefore, triangle ARE and BRD are similar, and $AR : BR = AE : BD = AF : BF$. If follows that RF bisects angle ARB, so that RF passes through the midpoint of minor arc AB on Γ. Similarly, PD and QE are respective bisectors of angles BPC and CQA and pass through the midpoints of the minor arc BC and CA on γ..

Let O be the centre of circle Γ, and U, V, W be the respective midpoints of the minor arc BC, CA, AB on this circle, so that PU contains D, QV contains E and RW contains F. It is required to prove that DU, EV and FW are concurrent.

Since ID and OU are perpendicular to BC, $ID \| OU$. Similarly, $IE \| OV$ and $IF \| OW$. Since $|ID| = |IE| = |IF| = r$ (the inradius) and $|OU| = |OV| = |OW| = R$ (the circumradius), a translation $\overrightarrow{IO}$ followed by a dilatation of factor R/r takes triangle DEF to triangle UVW, so that these triangles are similar with corresponding sides parallel.

Suppose that EV and FW intersect at K and that DU and FW intersect at L. Taking account of the similarity of the triangles KEF and KVW, LDF and LUW, DEF and UVW, we have that

$$KF : FW = EF : VW = DF : UW = LF : LW \ ,$$

so that $K = L$ and the lines DU, EV and FW intersect in a common point K, as desired.