Sun Life Financial

46th Canadian Mathematical Olympiad

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CMS SMC

Problems and Solutions

1. Let $a_1, a_2, \dots, a_n$ be positive real numbers whose product is 1. Show that the sum

$$\frac{a_1}{1+a_1}+\frac{a_2}{(1+a_1)(1+a_2)}+\frac{a_3}{(1+a_1)(1+a_2)(1+a_3)}+\cdots+\frac{a_n}{(1+a_1)(1+a_2)\cdots(1+a_n)}$$

is greater than or equal to $\dfrac{2^n-1}{2^n}$.

Solution. Note for that every positive integer m,

$$\begin{aligned}\frac{a_m}{(1+a_1)(1+a_2)\cdots(1+a_m)} &= \frac{1+a_m}{(1+a_1)(1+a_2)\cdots(1+a_m)}-\frac{1}{(1+a_1)(1+a_2)\cdots(1+a_m)}\\ &= \frac{1}{(1+a_1)\cdots(1+a_{m-1})}-\frac{1}{(1+a_1)\cdots(1+a_m)}.\end{aligned}$$

Therefore, if we let $b_j = (1+a_1)(1+a_2)\cdots(1+a_j)$, with $b_0 = 0$, then by telescoping sums,

$$\sum_{j=1}^{n}\frac{a_j}{(1+a_1)\cdots(1+a_j)}=\sum_{j=1}^{n}\left(\frac{1}{b_{j-1}}-\frac{1}{b_j}\right)=1-\frac{1}{b_n}.$$

Note that $b_n = (1+a_1)(1+a_2)\cdots(1+a_n) \geq (2\sqrt{a_1})(2\sqrt{a_2})\cdots(2\sqrt{a_n}) = 2^n$, with equality if and only if all a_i's equal to 1. Therefore,

$$1-\frac{1}{b_n}\geq 1-\frac{1}{2^n}=\frac{2^n-1}{2^n}.$$

To check that this minimum can be obtained, substituting all $a_i = 1$ to yield

$$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^n}=\frac{2^{n-1}+2^{n-2}+\ldots+1}{2^n}=\frac{2^n-1}{2^n},$$

as desired.

2. Let m and n be odd positive integers. Each square of an m by n board is coloured red or blue. A row is said to be red-dominated if there are more red squares than blue squares in the row. A column is said to be blue-dominated if there are more blue squares than red squares in the column. Determine the maximum possible value of the number of red-dominated rows plus the number of blue-dominated columns. Express your answer in terms of m and n.

Solution. The answer is $m+n-2$ if $m, n \geq 3$ and $\max\{m, n\}$ if one of m, n is equal to 1.

Note that it is not possible that all rows are red-dominated and all columns are blue-dominated. This is true, since the number of rows and columns are both odd, the number of squares is odd. Hence, there are more squares of one color than the other. Without loss of generality, suppose there are more red squares than blue squares. Then it is not possible that for every column, there are more blue squares than red squares. Hence, every column cannot be blue-dominated.

If one of m, n is equal to 1, say m without loss of generality, then by the claim, the answer is less than $n+1$. The example where there are n blue-dominated columns is by painting every square blue. There are 0 red-dominated rows. The sum of the two is $n = \max\{m, n\}$.

Now we handle the case $m, n \geq 3$.

There are m rows and n columns on the board. Hence, the answer is at most $m+n$. We have already shown that the answer cannot be $m+n$.

Since m, n are odd, let $m = 2a-1$ and $n = 2b-1$ for some positive integers a, b. Since $m, n \geq 3$, $a, b \geq 2$. We first show that the answer is not $m+n-1$. By symmetry, it suffices to show that we cannot have all rows red-dominated and all-but-one column blue-dominated. If all rows are red dominated, then each row has at least b red squares. Hence, there are at least $bm = (2a-1)b$ red squares. Since all-but-one column is blue-dominated, there are at least $2b-2$ blue-dominated columns. Each such column then has at least a blue squares. Therefore, there are at least $a(2b-2)$ blue squares. Therefore, the board has at least $(2a-1)b + a(2b-2) = 4ab - b - 2a$ squares. But the total number of squares on the board is

$$(2a-1)(2b-1) = 4ab - 2a - 2b + 1 = 4ab - 2a - b - b + 1 < 4ab - 2a - b,$$

which is true since $b \geq 2$. This is a contradiction. Therefore, the answer is less than $m+n-1$.

We now claim that there is a colouring of the board such that the number of blue-dominated columns plus the number of red-dominated rows is $m+n-2$; Colour the first column entirely red, and the first row, minus the top-left corner, entirely blue. The remaining uncoloured square is an even-by-even board. Colour the remaining board in an alternating pattern (i.e. checkerboard pattern). Hence, on this even-by-even board, each row has the same number of red squares as blue squares and each column has the same number of red squares as blue squares. Then on the whole board, since the top row, minus the top-left square is blue, all columns, but the leftmost column, are blue-dominated. Hence, there are $n-1$ blue-dominated columns. Since the left column is red, all rows but the top row are red dominated. Hence, there are $m-1$ red-dominated rows. The sum of these two quantities is $m+n-2$, as desired.

3. Let p be a fixed odd prime. A p-tuple $(a_1, a_2, a_3, \ldots, a_p)$ of integers is said to be *good* if

(i) $0 \leq a_i \leq p-1$ for all i, and

(ii) $a_1 + a_2 + a_3 + \cdots + a_p$ is not divisible by p, and

(iii) $a_1a_2 + a_2a_3 + a_3a_4 + \cdots + a_pa_1$ is divisible by p.

Determine the number of good p-tuples.

Solution. Let S be the set of all sequences $(b_1, b_2, \ldots, b_p)$ of numbers from the set $\{0, 1, 2, \ldots, p-1\}$ such that $b_1 + b_2 + \cdots + b_p$ is not divisible by p. We show that $|S| = p^p - p^{p-1}$. For let $b_1, b_2, \ldots, b_{p-1}$ be an arbitrary sequence of numbers chosen from $\{0, 1, 2, \ldots, p-1\}$. There are exactly $p-1$ choices for b_p such that $b_1 + b_2 + \cdots + b_{p-1} + b_p \not\equiv 0 \pmod{p}$, and therefore $|S| = p^{p-1}(p-1) = p^p - p^{p-1}$.

Now it will be shown that the number of good sequences in S is $\frac{1}{p}|S|$. For a sequence $B = (b_1, b_2, \ldots, b_p)$ in S, define the sequence $B_k = (a_1, a_2, \ldots, a_p)$ by

$$a_i = b_i - b_1 + k \bmod p$$

for $1 \leq i \leq p$. Now note that B in S implies that

$$a_1 + a_2 + \cdots + a_p \equiv (b_1 + b_2 + \cdots + b_p) - pb_1 + pk \equiv (b_1 + b_2 + \cdots + b_p) \not\equiv 0 \pmod{p}$$

and therefore B_k is in S for all non-negative k. Now note that B_k has first element k for all $0 \leq k \leq p-1$ and therefore the sequences $B_0, B_1, \ldots, B_{p-1}$ are distinct.

Now define the *cycle* of B as the set $\{B_0, B_1, \ldots, B_{p-1}\}$. Note that B is in its own cycle since $B = B_k$ where $k = b_1$. Now note that since every sequence in S is in exactly one cycle, S is the disjoint union of cycles.

Now it will be shown that exactly one sequence per cycle is good. Consider an arbitrary cycle $B_0, B_1, \ldots, B_{p-1}$, and let $B_0 = (b_1, b_2, \ldots, b_p)$ where $b_0 = 0$, and note that $B_k = (b_1 + k, b_2 + k, \ldots, b_p + k) \bmod p$. Let $u = b_1 + b_2 + \cdots + b_p$, and $v = b_1 b_2 + b_2 b_3 + \cdots + b_p b_1$ and note that $(b_1 + k)(b_2 + k) + (b_2 + k)(b_3 + k)) + \cdots + (b_p + k)(b_1 + k) = u + 2kv \bmod p$ for all $0 \le k \le p - 1$. Since $2v$ is not divisible by p, there is exactly one value of k with $0 \le k \le p - 1$ such that p divides $u + 2kv$ and it is exactly for this value of k that B_k is good. This shows that exactly one sequence per cycle is good and therefore that the number of good sequences in S is $\frac{1}{p}|S|$, which is $p^{p-1} - p^{p-2}$.

4. The quadrilateral $ABCD$ is inscribed in a circle. The point P lies in the interior of $ABCD$, and $\angle PAB = \angle PBC = \angle PCD = \angle PDA$. The lines AD and BC meet at Q, and the lines AB and CD meet at R. Prove that the lines PQ and PR form the same angle as the diagonals of $ABCD$.

Solution. . Let Γ be the circumcircle of quadrilateral $ABCD$. Let $\alpha = \angle PAB = \angle PBC \angle PCD = \angle PDA$ and let T_1 , T_2 , T_3 and T_4 denote the circumcircles of triangles APD, BPC, APB and CPD, respectively. Let M be the intersection of T_1 with line RP and let N be the intersection of T_3 with line SP. Also let X denote the intersection of diagonals AC and BD.

By power of a point for circles T_1 and Γ, it follows that $RM \cdot RP = RA \cdot RD = RB \cdot RC$ which implies that the quadrilateral $BMPC$ is cyclic and M lies on T_2. Therefore $\angle PMB = \angle PCB = \alpha = \angle PAB = \angle DMP$ where all angles are directed. This implies that M lies on the diagonal BD and also that $\angle XMP = \angle DMP = \alpha$. By a symmetric argument applied to S, T_3 and T_4, it follows that N lies on T_4 and that N lies on diagonal AC with $\angle XNP = \alpha$. Therefore $\angle XMP = \angle XNP$ and X, M, P and N are concyclic. This implies that the angle formed by lines MP and NP is equal to one of the angles formed by lines MX and NX. The fact that M lies on BD and RP and N lies on AC and SP now implies the desired result.

5. Fix positive integers n and $k \ge 2$. A list of n integers is written in a row on a blackboard. You can choose a contiguous block of integers, and I will either add 1 to all of them or subtract 1 from all of them. You can repeat this step as often as you like, possibly adapting your selections based on what I do. Prove that after a finite number of steps, you can reach a state where at least $n - k + 2$ of the numbers on the blackboard are all simultaneously divisible by k.

Solution. We will think of all numbers as being residues mod k. Consider the following strategy:

- If there are less than $k-1$ non-zero numbers, then stop.
- If the first number is 0, then recursively solve on the remaining numbers.
- If the first number is j with $0 < j < k$, then choose the interval stretching from the first number to the jth-last non-zero number.

First note that this strategy is indeed well defined. The first number must have value between 0 and $k-1$, and if we do not stop immediately, then there are at least $k-1$ non-zero numbers, and hence the third step can be performed.

For each j with $1 \leq j \leq k-2$, we claim the first number can take on the value of j at most a finite number of times without taking on the value of $j-1$ in between. If this were to fail, then every time the first number became j, I would have to add 1 to the selected numbers to avoid making it $j-1$. This will always increase the j-th last non-zero number, and that number will never be changed by other steps. Therefore, that number would eventually become 0, and the next last non-zero number would eventually become zero, and so on, until the first number itself becomes the j-th last non-zero number, at which point we are done since $j \leq k-2$.

Rephrasing slightly, if $1 \leq j \leq k-2$, the first number can take on the value of j at most a finite number of times between each time it takes on the value of $j-1$. It then immediately follows that if the first number can take on the value of $j-1$ at most a finite number of times, then it can also only take on the value of j a finite number of times. However, if it ever takes on the value of 0, we have already reduced the problem to $n-1$, so we can assume that never happens. It then follows that the first number can take on all the values $0, 1, 2, \ldots, k-2$ at most a finite number of times.

Finally, every time the first number takes on the value of $k-1$, it must subsequently take on the value of $k-2$ or 0, and so that can also happen only finitely many times.