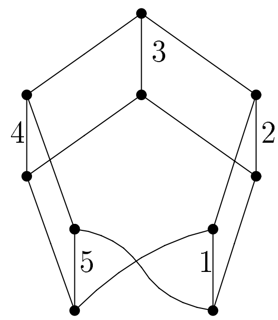
International Mathematical Talent Search – Round 18

Problem 1/18. Determine the minimum length of the interval $[a, b]$ such that $a \le x + y \le b$ for all real numbers $x \ge y \ge 0$ for which $19x + 95y = 1995$.

Problem 2/18. For a positive integer $n \ge 2$, let $P(n)$ denote the product of the positive integer divisors (including 1 and n) of n. Find the smallest n for which $P(n) = n^{10}$.

Problem 3/18. The graph shown on the right has 10 vertices, 15 edges, and each vertex is of order 3 (i.e., at each vertex 3 edges meet). Some of the edges are labeled 1, 2, 3, 4, 5 as shown. Prove that it is possible to label the remaining edges 6, 7, 8, $\ldots$, 15 so that at each vertex the sum of the labels on the edges meeting at that vertex is the same.



Problem 4/18. Let a, b, c, d be distinct real numbers such that

$$a + b + c + d = 3 \qquad \text{and} \qquad a^2 + b^2 + c^2 + d^2 = 45.$$

Find the value of the expression

$$\frac{a^5}{(a-b)(a-c)(a-d)} + \frac{b^5}{(b-a)(b-c)(b-d)} + \frac{c^5}{(c-a)(c-b)(c-d)} + \frac{d^5}{(d-a)(d-b)(d-c)}.$$

Problem 5/18. Let a and b be two lines in the plane, and let C be a point as shown in the figure on the right. Using only a compass and an unmarked straight edge, construct an isosceles right triangle ABC, so that A is on line a, B is on line b, and AB is the hypotenuse of $\triangle ABC$.