International Mathematical Talent Search – Round 40

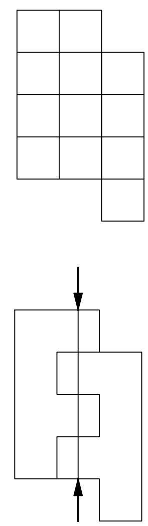
Problem 1/40. Determine all positive integers with the property that they are one more than the sum of the squares of their digits in base 10.

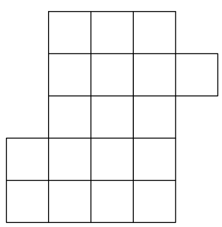
Problem 2/40. Prove that if n is an odd positive integer, then

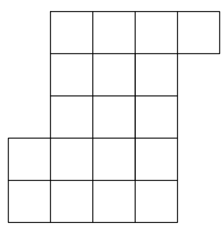
$$N = 2269^n + 1779^n + 1730^n - 1776^n$$

is an integer multiple of 2001.

Problem 3/40. The figure on the right can be divided into two congruent halves that are related to each other by a glide reflection, as shown below it. A glide reflection reflects a figure about a line, but also moves the reflected figure in a direction parallel to that line. For a square-grid figure, the only lines of reflection that keep its reflection on the grid are horizontal, vertical, $45°$ diagonal, and $135°$ diagonal. Of the two figures below, divide one figure into two congruent halves related by a glide reflection, and tell why the other figure cannot be divided like that.







Problem 4/40. Let A and B be points on a circle which are not diametrically opposite, and let C be the midpoint of the smaller arc between A and B. Let D, E and F be the points determined by the intersections of the tangent lines to the circle at A, B, and C. Prove that the area of $\triangle DEF$ is greater than half of the area of $\triangle ABC$.

Problem 5/40. Hexagon $RSTUVW$ is constructed by starting with a right triangle of legs measuring p and q, constructing squares outwardly on the sides of this triangle, and then connecting the outer vertices of the squares, as shown in the figure on the right.

Given that p and q are integers with $p > q$, and that the area of $RSTUVW$ is 1922, determine p and q.