

Solutions

49. Find all ordered pairs (x, y) that are solutions of the following system of two equations (where a is a parameter):

$$\begin{aligned} x - y &= 2 \\ \left(x - \frac{2}{a}\right)\left(y - \frac{2}{a}\right) &= a^2 - 1. \end{aligned}$$

Find all values of the parameter a for which the solutions of the system are two pairs of nonnegative numbers. Find the minimum value of $x + y$ for these values of a .

Solution. We need to assume that $a \neq 0$. Substitute $y = x - 2$ into the second equation to obtain

$$x^2 - 2x - \frac{4}{a}(x - 1) + \frac{4}{a^2} = a^2 - 1.$$

This can be manipulated to

$$\left[x - \left(1 + \frac{2}{a}\right)\right]^2 = a^2,$$

from which we find that

$$(x, y) = (1 + (2/a) + a, -1 + (2/a) + a), (1 + (2/a) - a, -1 + (2/a) - a).$$

For x and y to be nonnegative for both solutions, we require all of the four inequalities:

$$\begin{aligned} a(a^2 + a + 2) &\geq 0, & a(a^2 - a + 2) &\geq 0, \\ a(a - 2)r(a + 1) &= a(a^2 - a - 2) \leq 0, & a(a + 2)(a - 1) &= a(a^2 + a - 2) \leq 0. \end{aligned}$$

The last two equations entail that $a \leq -2$ or $0 < a \leq 1$. When $a \leq -2$, the first two inequalities fail to hold, while if $0 < a \leq 1$, all four equations hold. Thus, the solutions are two pairs of nonnegative numbers when $0 < a \leq 1$.

When both x and y are nonnegative, then $x + y \geq x - y = 2$. When $a = 1$, we obtain the solutions $(x, y) = (4, 2)$ and $(2, 0)$, and in the latter case, $x + y = 2$. Hence the minimum value of $x + y$ is 2.

Comment. The minimum of $x + y$ can be also obtained with more effort directly from the solutions in terms of a . Now $x + y = (4/a) \pm 2a$. Since

$$\left(\frac{4}{a} + 2a\right) - 6 = \frac{2(2 - a)(1 - a)}{a} \geq 0$$

for $0 < a \leq 1$, $(4/a) + 2a$ assumes its minimum value of 6 when $a = 1$. Since

$$\left(\frac{4}{a} - 2a\right) - 2 = \frac{2(2 + a)(1 - a)}{a} \geq 0$$

for $0 < a \leq 1$, $(4/a) - 2a$ assumes its minimum value of 2 when $a = 1$. Hence the minimum possible value of $x + y$ is 2.

50. Let n be a natural number exceeding 1, and let A_n be the set of *all* natural numbers that are not relatively prime with n (i.e., $A_n = \{x \in \mathbf{N} : \gcd(x, n) \neq 1\}$). Let us call the number n *magic* if for each two numbers $x, y \in A_n$, their sum $x + y$ is also an element of A_n (i.e., $x + y \in A_n$ for $x, y \in A_n$).

(a) Prove that 67 is a magic number.

(b) Prove that 2001 is **not** a magic number.

(c) Find all magic numbers.

Solution. [O. Bormashenko] (a) 67 is prime, so that all numbers that are not relatively prime with 67 and are elements of A_{67} are the multiples of 67. The sum of two such numbers is also a multiple of 67, and hence belongs to A_{67} . Therefore 67 is a magic number.

(b) $2001 = 3 \times 23 \times 29$. Now 3 and 23 belong to A_{2001} , but $26 = 3 + 23$ does not, because $\gcd(26, 2001)$ is equal to 1. Thus, 2001 is not a magic number.

(c) First, let us prove that all prime powers are magic numbers. Suppose that $n = p^k$ for some prime p and positive integer k . Then the numbers not relatively prime to n are precisely those that are divisible by p (as this is the only prime that can divide any divisor of n). The sum of any two such numbers is also divisible by p , so that A_n is closed under addition and n is magic.

Now suppose that n is not a power of a prime. Then $n = ab$, where a and b are two relatively prime numbers exceeding 1. Clearly, a and b belong to A_n . However, $\gcd(a, a+b) = \gcd(a, b) = 1$, and $\gcd(b, a+b) = \gcd(b, a) = 1$, so that $a+b$ is relatively prime to both a and b , and hence to $n = ab$. Thus, $a+b \notin A_n$, so that n is not magic.

We conclude that the set of prime powers equals the set of magic numbers.

51. In the triangle ABC , $AB = 15$, $BC = 13$ and $AC = 12$. Prove that, for this triangle, the angle bisector from A , the median from B and the altitude from C are concurrent (*i.e.*, meet in a common point).

Solution. [K. Ho] Denote by E the intersection of AC with the median from B (so that E is the midpoint of AC), by F the intersection of AB with the altitude from C (so that $CF \perp AB$), by D the intersection of BC and the bisector from A , and by x the length of AF . Then $BF = AB - AF = 15 - x$. By the angle-bisector theorem, $CD : DB = AC : AB = 12 : 15 = 4 : 5$. By the pythagorean theorem for the triangles AFC and BFC , $AC^2 - AF^2 = CF^2 = BC^2 - BF^2 \Leftrightarrow 12^2 - x^2 = 13^2 - (15 - x)^2 = -56 + 30x - x^2$. Thus, $x = 20/3$ and $15 - x = 25/3$, so that $AF : FB = 4 : 5$. Since $(AF/FB)(BD/DC)/(CE/EA) = (4/5)(5/4)1 = 1$, Ceva's theorem tells us that AD , BE and CF are concurrent.

52. One solution of the equation $2x^3 + ax^2 + bx + 8 = 0$ is $1 + \sqrt{3}$. Given that a and b are rational numbers, determine its other two solutions.

Solution 1. [R. Barrington Leigh] Since $1 + \sqrt{3}$ satisfies the given equation,

$$0 = 2(1 + \sqrt{3})^3 + a(1 + \sqrt{3})^2 + b(1 + \sqrt{3}) + 8 = (28 + 4a + b) + (12 + 2a + b)\sqrt{3}.$$

Since a and b are rational, this is possible only when $28 + 4a + b = 12 + 2a + b = 0$, or $(a, b) = (-8, 4)$. The equation is thus $0 = 2x^3 - 8x^2 + 4x + 8$. By inspection, we find that 2 is a root, and so the three roots turn out to be 2, $1 + \sqrt{3}$ and $1 - \sqrt{3}$.

Comment. Once 2 and $1 + \sqrt{3}$ are known to be roots, we can get the third root by noting that the sum of the roots is $-(-8)/2 = 4$.

Solution 2. A more structural way of getting the result is to note that the mapping that takes a surd $u + v\sqrt{3}$ (with u and v rational) to its surd conjugate $u - v\sqrt{3}$ preserves addition, subtraction and multiplication (*i.e.*, the surd conjugate of the sum (resp. product) of two surds is equal to the sum (resp. product) of the surd conjugates. The surd conjugate of a rational is the rational itself. Transforming all the elements of the equation to their surd conjugates, gives the same equation with x replaced by its surd conjugate. Thus, the surd conjugate $1 - \sqrt{3}$ of $1 + \sqrt{3}$ also satisfies the equation. The quadratic with these as roots is $x^2 - 2x - 2$, and this quadratic must be a factor of the given cubic. Since $2x^3 + ax^2 + bx + 8 = (x^2 - 2x - 2)(2 + [a + 4]x) + (2a + b + 12)x + (16 + 2a)$, we must have $2a + b + 12 = 16 + 2a = 0$, whence $(a, b) = (-8, 4)$, and

$$2x^3 + ax^2 + bx + 8 = 2(x^2 - 2x - 2)(x - 2)$$

and the third root is 2.

53. Prove that among any 17 natural numbers chosen from the sets $\{1, 2, 3, \dots, 24, 25\}$, it is always possible to find two whose product is a perfect square.

Solution. [K. Ho] Consider the following 16 sets: $\{1, 4, 9, 16, 25\}$, $\{2, 8, 18\}$, $\{3, 12\}$, $\{5, 20\}$, $\{6, 24\}$, $\{7\}$, $\{10\}$, $\{11\}$, $\{13\}$, $\{14\}$, $\{15\}$, $\{17\}$, $\{19\}$, $\{21\}$, $\{22\}$, $\{23\}$. In any of the sets with more than one element, the product of any two elements is a perfect square. Any choice of 17 numbers from among these sets must, by the Pigeonhole Principle, yield at least two from the same set. The product of these two must be a square. The desired result follows.

54. A circle has exactly one common point with each of the sides of a $(2n + 1)$ -sided polygon. None of the vertices of the polygon is a point of the circle. Prove that at least one of the sides is a tangent of the circle.

Solution. [J.Y. Jin] Assume that none of the sides is tangent to the circle. Let $A_1, A_2, \dots, A_{2n-1}$ be the consecutive vertices of a $(2n + 1)$ -sided polygon. Each side of the polygon has exactly one common point with the circle, none of the vertices lies on the circle and, by assumption, none of the sides is tangent to the circle. Therefore, for each side, one of the endpoints lies inside the circle and the second endpoint lies outside the circle. Colour the vertices inside the circle blue and the ones outside the circle red. Apparently, the colours of the consecutive vertices alternate (*i.e.*, blue, red, blue, red, etc.). Since there are $2n + 1$ vertices, A_1 and A_{2n+1} must have the same colour. However, these two vertices are the endpoints of a side of the polygon and so they cannot have the same colour. Thus, the assumption leads to a contradiction, and so at least one of the sides is tangent to the circle.