Solutions

31. Let x, y, z be positive real numbers for which $x^2 + y^2 + z^2 = 1$. Find the minimum value of

$$S = \frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y} \ .$$

Solution 1. [S. Niu] Let $a = yz/x$, $b = zx/y$ and $c = xy/z$. Then a, b, c are positive, and the problem becomes to minimize $S = a + b + c$ subject to $ab + bc + ca = 1$. Since

$$2(a^2 + b^2 + c^2 - ab - bc - ca) = (a-b)^2 + (b-c)^2 + (c-a)^2 \geq 0,$$

we have that $a^2 + b^2 + c^2 \geq ab + bc + ca$. Thus,

$$\begin{aligned} 1 &= ab + bc + ca \leq a^2 + b^2 + c^2 \\ &= (a+b+c)^2 - 2(ab+bc+ca) = (a+b+c)^2 - 2 = S^2 - 2 \end{aligned}$$

so $S \geq \sqrt{3}$ with equality if and only if $a = b = c$, or $x = y = z = 1/\sqrt{3}$. The desired result follows.

Solution 2. We have that

$$\begin{aligned} S^2 &= \frac{x^2y^2}{z^2} + \frac{y^2z^2}{x^2} + \frac{z^2x^2}{y^2} + 2(x^2+y^2+z^2) \\ &= \frac{1}{2}\left[\left(\frac{x^2y^2}{z^2} + \frac{z^2x^2}{y^2}\right) + \left(\frac{x^2y^2}{z^2} + \frac{y^2z^2}{x^2}\right) + \left(\frac{y^2z^2}{x^2} + \frac{z^2x^2}{y^2}\right)\right] + 2 \\ &\geq x^2 + y^2 + z^2 + 2 = 3 \end{aligned}$$

by the Arithmetic-Geometric Means Inequality. Equality holds if and only if $xy/z = yz/x = zx/y$, which is equivalent to $x = y = z$. Hence $S \geq \sqrt{3}$ if and only if $x = y = z = 1/\sqrt{3}$.

32. The segments BE and CF are altitudes of the acute triangle ABC, where E and F are points on the segments AC and AB, respectively. ABC is inscribed in the circle $\mathbf{Q}$ with centre O. Denote the orthocentre of ABC by H, and the midpoints of BC and AH be M and K, respectively. Let $\angle CAB = 45°$.

(a) Prove, that the quadrilateral $MEKF$ is a square.

(b) Prove that the midpoint of both diagonals of $MEKF$ is also the midpoint of the segment OH.

(c) Find the length of EF, if the radius of $\mathbf{Q}$ has length 1 unit.

Solution 1. (a) Since AH is the hypotenuse of right triangles AFH and AHE, $KF = KH = KA = KE$. Since BC is the hypotenuse of each of the right triangles BCF and BCE, we have that $MB = MF = ME = MC$. Since $\angle BAC = 45°$, triangles ABE, HFB and ACF are isosceles right triangles, so $\angle ACF = \angle ABE = \angle FBH = \angle FHB = 45°$ and $FA = FC$, $FH = FB$.

Consider a $90°$ rotation with centre F that takes $H \to B$. Then $FA \to FC$, $FH \to FB$, so $\Delta FHA \to \Delta FBC$ and $K \to M$. Hence $FK = FM$ and $\angle KFM = 90°$.

But $FK = KE$ and $FM = ME$, so $MEKF$ is an equilateral quadrilateral with one right angle, and hence is a square.

(b) Consider a $180°$ rotation (half-turn) about the centre of the square. It takes $K \leftrightarrow M$, $F \leftrightarrow E$ and $H \leftrightarrow H'$. By part (a), $\Delta FHA \equiv \Delta FBC$ and $AH \perp BC$. Since $KH \| MH'$ (by the half-turn), $MH' \perp BC$. Since $AH = BC$, $BM = \frac{1}{2}BC = \frac{1}{2}AH = KH = MH'$, so that BMH' is a right isosceles triangle and

$\angle CH'M = \angle BH'M = 45°$. Thus, $\angle BH'C = 90°$. Since $\angle BAC = 45°$, H' must be the centre of the circle through ABC. Hence $H' = O$. Since O is the image of H by a half-turn about the centre of the square, this centre is the midpoint of OH' as well as of the diagonals.

(c) $|EF| = \sqrt{2}|FM| = \sqrt{2}|BM| = |OB| = 1$.

Solution 2. [M. Holmes] (a) Consider a Cartesian plane with origin $F(0,0)$ and x−axis along the line AB. Let the vertices of the triangle be $A(-1,0)$, $C(0,1)$, $B(b,0)$. Since the triangle is acute, $0 < b < 1$. The point E is at the intersection of the line AC ($y = x + 1$) and a line through B with slope -1, so that $E = (\frac{1}{2}(b-1), \frac{1}{2}(b+1))$. H is the intersection point of the lines BE and CF, so H is at $(0,b)$; K is the midpoint of AH, so K is at $(-\frac{1}{2}, \frac{b}{2})$; M is the midpoint of BC, so M is at $(\frac{b}{2}, \frac{1}{2})$. It can be checked that the midpoints of EF and KM are both at $(\frac{1}{4}(b-1), \frac{1}{4}(b+1))$. The slope of EF is $(b+1)/(b-1)$ and that of KM is the negative reciprocal of this, so that $EF \perp KM$. It is straightforward to check that the lengths of EF and KM are equal, and we deduce that $EKFM$ is a square.

(b) O is the intersection point of the right bisectors of AB, AC and BC. The line $x + y = 0$ is the right bisector of AC and the abscissae of points on the right bisector of BC are all $\frac{1}{2}(b-1)$. Hence O is at $(\frac{1}{2}(b-1), \frac{1}{2}(1-b))$. It can be checked that the midpoint of OH agress with the joint midpoint of EF and KM.

(c) This can be checked by using the coordinates of points already identified.

Comment. One of the most interesting theorems in triangle geometry states that for each triangle there exists a circle that passes through the following nine special points: *the three midpoints of the sides*; *the three intersections of sides and altitudes (pedal points); and the three midpoints of the segments connecting the vertices to the orthocentre*. This circle is called the *nine-point circle*. If H is the orthocentre and O is the circumcentre, then the centre of the nine-point circle is the midpoint of OH. Note that in this problem, the points M, E, F, K belong to the nine-point circle.

33. Prove the inequality $a^2 + b^2 + c^2 + 2abc < 2$, if the numbers a, b, c are the lengths of the sides of a triangle with perimeter 2.

Solution 1. Let $u = b + c - a$, $v = c + a - b$ and $w = a + b - c$, so that $2a = v + w$, $2b = u + w$ and $2c = u + v$. Then u, v, w are all positive and $u + v + w = 2$. The difference of the right and left sides multiplied by 4 is equal to

$$\begin{aligned}
&4[2 - (a^2 + b^2 + c^2 + 2abc)] \\
&\quad = 8 - (v + w)^2 - (u + w)^2 - (u + v)^2 - (v + w)(u + w)(u + v)] \\
&\quad = 8 - (u + v + w)^2 - (u^2 + v^2 + w^2) - (2 - u)(2 - v)(2 - w) \\
&\quad = 8 - 4 - (u^2 + v^2 + w^2) - 8 + 4(u + v + w) - 2(vw + uw + uv) + uvw \\
&\quad = 4 - (u + v + w)^2 - 8 + 4 \times 2 + uvw \\
&\quad = 4 - 4 - 8 + 8 + uvw = uvw > 0
\end{aligned}$$

as desired.

Solution 2. [L. Hong] The perimeter of the triangle is $a + b + c = 2$. We have that

$$\begin{aligned}
a^2 + b^2 + c^2 + 2abc &= (a + b + c)^2 + 2(a - 1)(b - 1)(c - 1) + 2 - 2(a + b + c) \\
&= 4 + 2(a - 1)(b - 1)(c - 1) + 2 - 4 = 2(a - 1)(b - 1)(c - 1) + 2\ .
\end{aligned}$$

Since $a < b + c$, $b < c + a$ and $c < a + b$, it follows that $a < 1$, $b < 1$, $c < 1$, from which the result follows.

34. Each of the edges of a cube is 1 unit in length, and is divided by two points into three equal parts. Denote by $\mathbf{K}$ the solid with vertices at these points.

(a) Find the volume of **K**.

(b) Every pair of vertices of **K** is connected by a segment. Some of the segments are coloured. Prove that it is always possible to find two vertices which are endpoints of the same number of coloured segments.

Solution. (a) The solid figure is obtained by slicing off from each corner a small tetrahedron, three of whose faces are pairwise mutually perpendicular at one vertex; the edges emanating from that vertex all have length $1/3$, and so the volume of each tetrahedron removed is $1/3(1/2 \cdot 1/3 \cdot 1/3)(1/3) = 1/162$. Since there are eight such tetrahedra removed, the volume of the resulting solid is $1-4/81 = 77/81$. (The numbers of vertices, edges and faces of the solid are respectively 24, 36 and 14.)

(b) The polyhedron has $3 \times 8 = 24$ vertices. Each edge from a given vertex is joined to 23 vertices. The possible number of coloured segments emanating from a vertex is one of the twenty-four numbers, 0, 1, 2, $\cdots$, 23. But it is not possible for one vertex to be joined to all 23 others and another vertex to be joined to no other vertex. So there are in effect only 23 options for the number of coloured segments emanating from each of the 24 vertices. By the Pigeonhole Principle, there must be two vertices with the same number of coloured segments emanating from it.

35. There are n points on a circle whose radius is 1 unit. What is the greatest number of segments between two of them, whose length exceeds $\sqrt{3}$?

Solution. [O.Bormashenko] The side of the equilateral triangle inscribed in a circle of unit radius is $\sqrt{3}$. So the segment with length $\sqrt{3}$ is a chord subtending an angle of $120°$ at the centre. Therefore, there is no triangle with three vertices on the circle each of whose sides are longer than $\sqrt{3}$. Consider the graph whose vertices are all n given points and whose arcs all have segments longer than $\sqrt{3}$. This graph contains no triangles.

Recall Turan's theorem (see the solution of problem 23 in *Olymon 1:4*: *Let G be a graph with n vertices. Denote by $l(G)$ the number of its edges and $t(G)$ the number of triangles contained in the graph. If $t(G) = 0$, then $l(G) \leq \lfloor n^2/4 \rfloor$.* ¿From this theorem, it follows that the number of segments with chords exceeding $\sqrt{3}$ is at most $\lfloor n^2/4 \rfloor$.

To show that this maximum number can be obtained, first construct points A, B, C, D on the circle, so that the disjoint arcs AB and CD subtend angles of $120°$ at the centre. If $n = 2k+1$ is odd, place k points on the arc BC and $k+1$ points on the arc DA. Any segment containing a point in BC to a point in DA must subtend an angle exceeding $120°$, so its length exceeds $\sqrt{3}$. There are exactly $k(k+1) = \lfloor (2k+1)^2/4 \rfloor$ such segments. If $n = 2k$ is even, place k points in each of the arcs BC and CA, so that there are exactly $k^2 = \lfloor (2k)^2/4 \rfloor$ such segments. In either case, the maximum number of segments whose length exceeds $\sqrt{3}$ is $\lfloor n^2/4 \rfloor$.

36. Prove that there are not three rational numbers x, y, z such that

$$x^2 + y^2 + z^2 + 3(x + y + z) + 5 = 0 \ .$$

Solution. Suppose the $x = u/m$, $y = v/m$ and $z = w/m$, where m is the least common multiple of the denominators of x, y and z. Then, multiplying the given equation by m^2 yields

$$u^2 + v^2 + w^2 + 3(um + vm + wm) + 5m^2 = 0 \ .$$

A further multiplication by 4 and a rearrangement of terms yields

$$(2u + 3m)^2 + (2v + 3m)^2 + (2w + 3m)^2 = 7m^2 \ .$$

This is of the form

$$p^2 + q^2 + r^2 = 7n^2 \qquad (*)$$

for integers p, q, r, n. Suppose that n is even. Then, considering the equation modulo 4, we deduce that p, q, r must also be even. We can divide the equation by 4 to obtain another equation of the form (*) with smaller numbers. We can continue to do this as long as the resulting n turns out to be even. Eventually, we arrive at an equation for which n is odd, so that the right side is congruent to 7 modulo 8. But there is not combination of integers p, q and r for which $p^2 + q^2 + r^2 \equiv 7 \pmod 8$, so that the equation is impossible.