## Solutions

115. Let $U$ be a set of $n$ distinct real numbers and let $V$ be the set of all sums of distinct pairs of them, *i.e.*,

$$V = \{x + y : x, y \in U, x \neq y\} \ .$$

What is the smallest possible number of distinct elements that $V$ can contain?

*Solution.* Let $U = \{x_i : 1 \leq i \leq n\}$ and $x_1 < x_2 < \cdots < x_n$. Then

$$x_1 + x_2 < x_1 + x_3 < \cdots < x_1 + x_n < x_2 + x_n < \cdots < x_{n-1} + x_n$$

so that the $2n-3$ sums $x_1 + x_j$ with $2 \leq j \leq n$ and $x_i + x_n$ with $2 \leq i \leq n-1$ have distinct values. On the other hand, the set $\{1, 2, 3, 4, \cdots, n\}$ has the smallest pairwise sum $3 = 1 + 2$ and the largest pairwise sum $2n - 1 = (n-1) + n$, so there are at most $2n-3$ pairwise sums. Hence $V$ can have as few as, but no fewer than, $2n-3$ elements.

*Comment.* This problem was not well done. A set is assumed to be given, and so you must deal with it. Many of you tried to vary the elements in the set, and say that if we fiddle with them to get an arithmetic progression we minimize the number of sums. This is vague and intuitive, does not deal with the given set and needs to be sharpened. To avoid this, the best strategy is to take the given set and try to determine pairwise sums that are sure to be distinct, regardless of what the set is; this suggests that you should look at extreme elements - the largest and the smallest. To make the exposition straightforward, assume with no loss of generality that the elements are in increasing or decreasing order. You want to avoid a proliferation of possibilities.

Secondly, in setting out the proof, note that it should divide cleanly into two parts. First show that, whatever the set, at least $2n-3$ distinct sums occur. Then, by an example, demonstrate that exactly $2n-3$ sums are possible. A lot of solvers got into hot water by combining these two steps.

116. Prove that the equation

$$x^4 + 5x^3 + 6x^2 - 4x - 16 = 0$$

has exactly two real solutions.

*Solution.* In what follows, we denote the given polynomial by $p(x)$.

*Solution 1.*

$$p(x) = (x^2 + 3x + 4)(x^2 + 2x - 4) = \left(\left(x + \frac{3}{2}\right)^2 + \frac{7}{4}\right)\left((x+1)^2 - 5\right) \ .$$

The first quadratic factor has nonreal roots, and the second two real roots, and the result follows.

*Solution 2.* Since $p(1) = p(-2) = -8$, the polynomial $p(x) + 8$ is divisible by $x + 2$ and $x - 1$. We find that

$$p(x) = (x+2)^3(x-1) - 8 \ .$$

When $x > 1$, the linear factors are strictly increasing, so $p(x)$ strictly increases from $-8$ unboundedly, and so vanishes exactly once in the interval $(1, \infty)$. When $x < -2$, the two linear factors are both negative and increasing, so that $p(x)$ strictly decreases from positive values to $-8$. Thus, it vanishes exactly once in the interval $(-\infty, -2)$. When $-2 < x < 1$, the two linear factors have opposite signs, so that $(x+2)^3(x-1) < 0$ and $p(x) < -8 < 0$. The result follows.

*Solution 3.* [R. Mong] We have that $p(x) = x^4 + 5x^3 + 6x^2 - 4x - 16 = (x-1)(x+2)^3 - 8$. Let $q(x) = f(-(x+2)) = (-x-3)(-x)^3 - 8 = x^4 + 3x^3 - 8$. By Descartes' Rule of Signs, $p(x)$ and $q(x)$ each have exactly one positive root. (The rule says that the number of positive roots of a real polynomial has the same parity as and no more than the number of sign changes in the coefficients as read in descending order.) It follows that $p(x)$ has exactly one root in each interval $(-\infty,-2)$ and $(0,\infty)$. Since $p(x) \leq -8$ for $-2 \leq x \leq 0$, the desired result follows.

*Solution 4.* Since the derivative $p'(x) = (x+2)^2(4x-1)$, we deduce that $p'(x) < 0$ for $x < \frac{1}{4}$ and $p'(x) > 0$ for $x > \frac{1}{4}$. It follows that $p(x)$ is strictly decreasing on $(\infty, \frac{1}{4})$ and strictly increasing on $(\frac{1}{4}, \infty)$. Since the leading coefficient is positive and $p(\frac{1}{4}) < 0$, $p(x)$ has exactly one root in each of the two intervals.

117. Let $a$ be a real number. Solve the equation

$$(a-1)\left(\frac{1}{\sin x} + \frac{1}{\cos x} + \frac{1}{\sin x \cos x}\right) = 2 \ .$$

*Solution. First step.* When $a = 1$, the equation is always false and there is no solution. Also, the left side is undefined when $x$ is a multiple of $\pi/2$, so we exclude this possibility. Thus, in what follows, we suppose that $a \neq 1$ and that $\sin x \cos x \neq 0$. [*Comment.* This initial clearing away the underbrush avoids nuisance situations later and makes the exposition of the core of the solution go easier.]

*Solution 1.* Let $u = \sin x + \cos x$. Then $u^2 - 1 = 2\sin x \cos x$, so that

$$(a-1)(u+1) = u^2 - 1 \Longleftrightarrow$$

$$0 = u^2 - (a-1)u - a = (u+1)(u-a) \ .$$

Since $\sin x \cos x \neq 0$, $u + 1 \neq 0$. Thus, $u = a$, and $\sin x$, $\cos x$ are the roots of the quadratic equation

$$t^2 - at + \frac{a^2-1}{2} = 0 \ .$$

Hence

$$(\sin x, \cos x) = \left(\frac{1}{2}(a \pm \sqrt{2-a^2}), \frac{1}{2}(a \mp \sqrt{2-a^2})\right) \ .$$

For this to be viable, we require that $|a| \leq \sqrt{2}$ and $a \neq 1$.

*Solution 2.* The given equation (since $\sin x \cos x \neq 0$) is equivalent to

$$(a-1)(\sin x + \cos x + 1) = 2\sin x \cos x$$

$$\Longrightarrow (a-1)^2(2 + 2\sin x + 2\cos x + 2\sin x \cos x) = 4\sin^2 x \cos^2 x$$

$$\Longleftrightarrow 4(a-1)(\sin x \cos x) + 2(a-1)^2(\sin x \cos x) = 4\sin^2 x \cos^2 x$$

$$\Longleftrightarrow 2(a-1) + (a-1)^2 = \sin 2x$$

$$\Longleftrightarrow \sin 2x = a^2 - 1 \ .$$

For this to be viable, we require that $|a| \leq \sqrt{2}$.

For all values of $x$, we have that

$$2(1 + \sin x + \cos x) + 2\sin x \cos x = (\sin x + \cos x + 1)^2 \ ,$$

so that

$$(\sin x + \cos x + 1 - 1)^2 = 1 + 2\sin x \cos x = a^2 \ ,$$

whence

$$\sin x + \cos x = \pm a \ .$$

Since we squared the given equation, we may have introduced extraneous roots, so we need to check the solution. Taking $\sin x + \cos x = a$, we find that

$$(a-1)(\sin x + \cos x + 1) = (a-1)(a+1) = a^2 - 1 = 2\sin x \cos x$$

as desired. Taking $\sin x + \cos x = -a$, we find that

$$(a-1)(\sin x + \cos x + 1) = (a-1)(1-a) = -(a-1)^2 \neq a^2 - 1 = 2\sin x \cos x$$

so this does not work. Hence the equation is solvable when $|a| \leq \sqrt{2}$, $a \neq 1$, and the solution is given by $x = \frac{1}{2}\theta$ where $\sin\theta = a^2 - 1$ and $\sin x + \cos x = a$.

*Solution 3.* We have that

$$\begin{aligned}
(a-1)\left(\sqrt{2}\cos\left(x - \frac{\pi}{4}\right) + 1\right) &= (a-1)(\sin x + \cos x + 1) \\
&= 2\sin x \cos x = \sin 2x \\
&= \cos\left(\frac{\pi}{2} - 2x\right) \\
&= \cos 2\left(x - \frac{\pi}{4}\right) \\
&= 2\cos^2\left(x - \frac{\pi}{4}\right) - 1 \quad .
\end{aligned}$$

Let $t = \cos(x - \pi/4)$. Then

$$\begin{aligned}
0 &= 2t^2 - \sqrt{2}(a-1)t - a \\
&= (\sqrt{2}t - a)(\sqrt{2}t + 1) \ .
\end{aligned}$$

Since $x$ cannot be a multiple of $\pi/2$, $t$ cannot equal $1/\sqrt{2}$. Hence $\cos(x + \frac{\pi}{4}) = t = a/\sqrt{2}$, so that $x = \frac{\pi}{4} + \phi$ where $\cos\phi = a/\sqrt{2}$. Since the equation at the beginning of this solution is equivalent to the given equation and the quadratic in $t$, this solution is valid, subject to $|a| \leq \sqrt{2}$ and $a \neq 1$.

*Solution 4.* Note that $\sin x + \cos x = 1$ implies that $2\sin x \cos x = 0$. Since we are assuming that $\sin x + \cos x \neq 0$, we multiply the equation $(a-1)(\sin x + \cos x + 1) = 2\sin x \cos x$ by $\sin x + \cos x - 1$ to obtain the equivalent equation

$$\begin{aligned}
&(a-1)[(\sin x + \cos x)^2 - 1] = 2\sin x \cos x(\sin x + \cos x - 1) \\
&\iff (a-1)2\sin x \cos x = 2\sin x \cos x(\sin x + \cos x - 1) \\
&\iff \sin x + \cos x = a \\
&\iff \sin\left(x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{a}} \\
&\iff x = \theta - \frac{\pi}{4}
\end{aligned}$$

where $\sin\theta = a/\sqrt{2}$. We have the same conditions on $a$ as before.

*Solution 5.* The equation is equivalent to

$$(a-1)(\sin x + 1) = \cos x(2\sin x + 1 - a) \ .$$

Squaring, we obtain

$$(a-1)^2(\sin x + 1)^2 = (1 - \sin^2 x)[4\sin^2 x + 4(1-a)\sin x + (1-a)^2] \ .$$

Dividing by $\sin x + 1$ yields

$$2\sin^2 x - 2a\sin x + (a^2 - 1) = 0 \Longrightarrow \sin x = \frac{1}{2}(a \pm \sqrt{2 - a^2}) \ .$$

[Note that the equation $\cos x = a - \sin x$ in Solution 3 leads to the equation here.] Thus

$$\sin^2 x = \frac{1 \pm a\sqrt{2 - a^2}}{2} \qquad \text{and} \qquad \cos^2 x = \frac{1 \mp a\sqrt{2 - a^2}}{2} \ .$$

Thus

$$(\sin x, \cos x) = \left(\frac{1}{2}(a \pm \sqrt{2 - a^2}), \pm\frac{1}{2}(a \mp \sqrt{2 - a^2})\right) \ .$$

Note that $0 \leq \sin^2 x \leq 1$ requires $0 \leq 1 \pm a\sqrt{2 - a^2} \leq 2$, or equivalently $|a| \leq \sqrt{2}$ and $a^2(2 - a^2) \leq 1 \Leftrightarrow |a| \leq \sqrt{2}$ and $(a^2 - 1)^2 \geq 0 \Leftrightarrow |a| \leq \sqrt{2}$.

We need to check for extraneous roots. If

$$(\sin x, \cos x) = ((1/2)(a \pm \sqrt{2 - a^2}), (1/2)(a \mp \sqrt{2 - a^2}) \ ,$$

then

$$(a - 1)(\sin x + \cos x + 1) = (a - 1)(a + 1) = a^2 - 1 = (1/2)[a^2 - (2 - a^2)] = 2\sin x \cos x$$

as desired. On the other hand, if

$$(\sin x, \cos x) = ((1/2)(a \pm \sqrt{2 - a^2}), -(1/2)(a \mp \sqrt{2 - a^2}) \ ,$$

then

$$(a - 1)(\sin x + \cos x + 1) = (a - 1)(\sqrt{2 - a^2} + 1)$$

while

$$2\sin x \cos x = -(1/2)[a^2 - (2 - a^2)] = -(a^2 - 1) \ .$$

These are not equal when $a \neq 1$. Hence

$$(\sin x, \cos x) = ((1/2)(a \pm \sqrt{2 - a^2}), (1/2)(a \mp \sqrt{2 - a^2}) \ .$$

*Solution 6.* Let $u = \sin x + \cos x$, so that $u^2 = 1 + 2\sin x \cos x$. Then the equation is equivalent to $(a - 1)(u + 1) = u^2 - 1$, whence $u = 1$ or $u = a$. We reject $u = 1$, so that $u = a$ and we can finish as in Solution 3.

*Solution 7.* [O. Bormashenko] Since

$$\frac{1}{\sin x} + \frac{1}{\cos x} + \frac{1}{\sin x \cos x} = \frac{2}{a - 1} \ ,$$

we have that

$$\left(\frac{1}{\sin x} + \frac{1}{\cos x}\right)^2 = \left(\frac{2}{a - 1} - \frac{1}{\sin x \cos x}\right)^2$$

$$\Leftrightarrow \frac{1}{\sin^2 x \cos^2 x} + \frac{2}{\sin x \cos x} = \frac{4}{(a - 1)^2} - \frac{4}{(a - 1)\sin x \cos x} + \frac{1}{\sin^2 x \cos^2 x}$$

$$\Leftrightarrow \frac{1}{\sin x \cos x}\left(2 + \frac{4}{a - 1}\right) = \frac{4}{(a - 1)^2}$$

$$\Leftrightarrow 2\sin x \cos x = a^2 - 1 \Leftrightarrow \sin 2x = a^2 - 1 \ .$$

We check this solution as in Solution 2.

*Solution 8.* [S. Patel] Let $z = \cos x + i \sin x$. Note that $z \neq 0, \pm 1, \pm i$. Then

$$\sin x = \frac{1}{2i}\left(z - \frac{1}{z}\right) = \frac{z^2-1}{2iz}$$

and

$$\cos x = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2+1}{2z} \ .$$

The given equation is equivalent to

$$\begin{aligned} 1 &= (a-1)\left[\frac{iz}{z^2-1} + \frac{z}{z^2+1} + \frac{2iz^2}{z^4-1}\right] \\ &= (a-1)\left[\frac{iz}{z^2-1} + \frac{z}{z^2+1} + \frac{i}{z^2-1} + \frac{i}{z^2+1}\right] \\ &= (a-1)\left[\frac{i(z+1)}{z^2-1} + \frac{z+i}{z^2+1}\right] \\ &= (a-1)\left[\frac{i}{z-1} + \frac{1}{z-i}\right] \\ &= (a-1)\left[\frac{(i+1)z}{z^2-(1+i)z+i}\right] \ . \end{aligned}$$

This is equivalent to

$$z^2 - (1+i)z + i = (a-1)[(1+i)z] \Leftrightarrow z^2 - (1+i)az + i = 0 \ .$$

Hence

$$\begin{aligned} z &= \frac{(1+i)a \pm \sqrt{2ia^2-4i}}{2} \\ &= \frac{(1+i)a \pm \sqrt{2i}\sqrt{a^2-2}}{2} \\ &= \left(\frac{1+i}{2}\right)(a \pm \sqrt{a^2-2}) \ . \end{aligned}$$

Suppose that $a^2 > 2$. Then $|z|^2 = \frac{1}{2}[a \pm \sqrt{a^2-2}]^2 = (a^2-1) \pm a\sqrt{a^2-2}$. Since $|z| = 1$, we must have $a^2 - 2 = \pm a\sqrt{a^2-2}$, whence $a^4 - 4a^2 + 4 = a^4 - 2a^2$ or $a^2 = 2$, which we do not have. Hence, we must have $a^2 \leq 2$, so that

$$z = \left(\frac{1+i}{2}\right)(a \pm \sqrt{a^2-2}) \ .$$

Therefore

$$\cos x = Rez = \frac{a \mp \sqrt{2-a^2}}{2}$$

and

$$\sin x = Imz = \frac{a \pm \sqrt{2-a^2}}{2} \ .$$

*Comment.* R. Barrington Leigh had an interesting approach for solutions with positive values of $\sin x$ and $\cos x$. Consider a right triangle with legs $\sin x$ and $\cos x$, inradius $r$, semiperimeter $s$ and area $\Delta$. Then

$$\frac{1}{r} = \frac{s}{\Delta} = \frac{1 + \sin x + \cos x}{\sin x \cos x} = \frac{2}{a-1}$$

so that $1 \leq a$. We need to determine right triangles whose inradius is $\frac{1}{2}(a-1)$. Using the formula $r = (s-c)\tan(C/2)$ with $c = 1$ and $C = 90°$, we have that

$$r = (s-1)\tan 45° = s - 1$$

whence

$$\frac{a-1}{2} = \frac{1}{2}(\sin x + \cos x - 1) \Leftrightarrow \sin x + \cos x = a \ .$$

118. Let $a, b, c$ be nonnegative real numbers. Prove that

$$a^2(b+c-a) + b^2(c+a-b) + c^2(a+b-c) \leq 3abc \ .$$

When does equality hold?

*Solution 1.* Observe that

$$3abc - [a^2(b+c-a) + b^2(c+a-b) + c^2(a+b-c)] = abc - (b+c-a)(c+a-b)(a+b-c) \ . \qquad (*)$$

Now

$$2a = (c+a-b) + (a+b-c)$$

with similar equations for $b$ and $c$. These equations assure us that at most one of $b+c-a$, $c+a-b$ and $a+b-c$ can be negative. If exactly one of these three quantities is negative, than (*) is clearly nonnegative, and is equal to zero if and only if at least one of $a$, $b$ and $c$ vanishes, and the other two are equal. If all the three quantities are nonnegative, then an application of the arithmetic-geometric means inequality yields that

$$2a \geq 2\sqrt{(c+a-b)(a+b-c)}$$

with similar inequalities for $b$ and $c$. It follows from this that $(*)$ is nonnegative and vanishes if and only if $a = b = c$, or one of $a$, $b$, $c$ vanishes and the other two are equal.

*Solution 2.* Wolog, suppose that $a \leq b \leq c$. Then

$$\begin{aligned}
&3abc - [a^2(b+c-a) + b^2(c+a-b) + c^2(a+b-c)] \\
&\quad = a(bc - ab - ac + a^2) + b(ac - bc - ab + b^2) + c(ab - ac - bc + c^2) \\
&\quad = a(b-a)(c-a) - b(c-b)(b-a) - c(c-a)(c-b) \\
&\quad = a(b-a)(c-a) + (c-b)[ab - b^2 + c^2 - ca] \\
&\quad = a(b-a)(c-a) + (c-b)^2(c+b-a) \geq 0 \ .
\end{aligned}$$

Equality occurs if and only if $a = b = c$ or if $a = 0$ and $b = c$.

*Solution 3.* [S.-E. Lu] The inequality is equivalent to

$$(a+b-c)(b+c-a)(c+a-b) \leq abc \ .$$

At most one of the three factors on the left side can be negative. If one of them is negative, then the inequality is satisfied, and equality occurs if and only if both sides vanish (*i.e.*, one of the three variables vanishes and the others are equal).

Otherwise, we can square both sides to get the equivalent inequality:

$$[a^2 - (b-c)^2][b^2 - (a-c)^2][c^2 - (a-b)^2] \leq a^2b^2c^2 \ .$$

Since $b \leq a + c$ and $c \leq a + b$, we find that $|b - c| \leq a$, whence $(b-c)^2 \leq a^2$. Thus, $a^2 - (b-c)^2 \leq a^2$, with similar inequalties for the other two factors on the left. It follows that the inequality holds with equality when $a = b = c$ or one variable vanishes and the other two are equal.

*Solution 4.* [R. Mong] Let $\sum f(a,b,c)$ denote the cyclic sum $f(a,b,c)+f(b,c,a)+f(c,a,b)$. Suppose that $u = a^3+b^3+c^3 = \sum a^3$ and $v = \sum (b-c)a^2$. Then

$$u+v = \sum a^3 + (b-c)a^2 = \sum a^2(a+b-c)$$

and

$$u-v = \sum a^3 - (b-c)a^2 = \sum a^2(a-b+c) = \sum a^2(c+a-b) = \sum b^2(a+b-c) \ .$$

Then

$$\sum a^2(a+b-c)\sum b^2(a+b-c) = u^2 - v^2 \leq u^2 \ .$$

By the Cauchy-Schwarz Inequality,

$$\sum ab(a+b-c) \leq \sqrt{u^2-v^2} \leq a^3+b^3+c^3 \ .$$

(Note that the left side turns out to be positive; however, the result would hold anyway even if it were negative, since $a^3+b^3+c^3$ is nonnegative.)

Then

$$a^2b + ab^2 - abc + b^2c + bc^2 - abc + a^2c + ac^2 - abc \leq a^3+b^3+c^3$$
$$\Longrightarrow a^2(b+c-a) + b^2(c+a-b) + c^2(a+b-c) \leq 3abc$$

as desired.

Equality holds if and only if

$$a^2(a+b-c) : b^2(b+c-a) : c^2(c+a-b) = b^2(a+b-c) : c^2(b+c-a) : a^2(c+a-b) \ .$$

Suppose, if possible that $a+b=c$, say. Then $b+c-a=2b$, $c+a-b=2a$, so that $b^3 : c^2a = c^2b : a^3$ and $c^2 = ab$. This is possible if and only if $a=b=0$. Otherwise, all terms in brackets are nonzero, and we find that $a^2 : b^2 : c^2 = b^2 : c^2 : a^2$ so that $a=b=c$.

*Solution 5.* [A. Chan] Wolog, let $a \leq b \leq c$, so that $b = a+x$ and $c = a+x+y$, where $x, y \geq 0$. The left side of the inequality is equal to

$$3a^3 + 3(2x+y)a^2 + (2x^2+2xy-y^2)a - (y^3+2xy^2)$$

and the right side is equal to

$$3a^3 + 3(2x+y)a^2 + (3x^2+3xy)a \ .$$

The right side minus the left side is equal to

$$(x^2+xy+y^2)a + (y^3+2xy^2) \ .$$

Since each of $a, x, y$ is nonnegative, this expression is nonnegative, and it vanishes if and only if each term vanishes. Hence, the desired inequality holds, with equality, if and only if $y=0$ and either $a=0$ or $x=0$, if and only if either ($a=0$ and $b=c$) or ($a=b=c$).

119. The medians of a triangle $ABC$ intersect in $G$. Prove that

$$|AB|^2 + |BC|^2 + |CA|^2 = 3(|GA|^2 + |GB|^2 + |GC|^2) \ .$$

*Solution 1.* Let the respective lengths of $BC$, $CA$, $AB$, $AG$, $BG$ and $CG$ be $a, b, c, u, v, w$. If $M$ is the midpoint of $BC$, then $A, G, M$ are collinear with $AM = (3/2)AG$. Let $\theta = \angle AMB$. By the law of cosines, we have that

$$c^2 = \frac{9}{4}u^2 + \frac{1}{4}a^2 - \frac{3}{2}au\cos\theta$$

$$b^2 = \frac{9}{4}u^2 + \frac{1}{4}a^2 + \frac{3}{2}au\cos\theta$$

whence

$$b^2 + c^2 = \frac{9}{2}u^2 + \frac{1}{2}a^2 \ .$$

Combining this with two similar equations for the other vertices and opposite sides, we find that

$$2(a^2 + b^2 + c^2) = \frac{9}{2}(u^2 + v^2 + w^2) + \frac{1}{2}(a^2 + b^2 + c^2)$$

which simplifies to $a^2 + b^2 + c^2 = 3(u^2 + v^2 + w^2)$, as desired.

*Solution 2.* We have that

$$\overrightarrow{AG} = \frac{1}{3}(\overrightarrow{AB} + \overrightarrow{AC}) \ .$$

Hence

$$|\overrightarrow{AG}|^2 = \frac{1}{9}[|\overrightarrow{AB}|^2 + |\overrightarrow{AC}|^2 + 2(\overrightarrow{AB} \cdot \overrightarrow{AC})] \ .$$

Similarly

$$|\overrightarrow{BG}|^2 = \frac{1}{9}[|\overrightarrow{BA}|^2 + |\overrightarrow{BC}|^2 + 2(\overrightarrow{BA} \cdot \overrightarrow{BC})] \ ,$$

and

$$|\overrightarrow{CG}|^2 = \frac{1}{9}[|\overrightarrow{CA}|^2 + |\overrightarrow{CB}|^2 + 2(\overrightarrow{CA} \cdot \overrightarrow{CB})] \ .$$

Therefore

$$\begin{aligned} &9[|\overrightarrow{A}G|^2 + |\overrightarrow{BG}|^2 + |\overrightarrow{CG}|^2] \\ &= 2[|\overrightarrow{AB}|^2 + |\overrightarrow{BC}|^2 + |\overrightarrow{CA}|^2] + \overrightarrow{AB} \cdot (\overrightarrow{AC} + \overrightarrow{CB}) + \overrightarrow{AC} \cdot (\overrightarrow{AB} + \overrightarrow{BC}) + \overrightarrow{BC} \cdot (\overrightarrow{BA} + \overrightarrow{AC}) \\ &= 3[|\overrightarrow{AB}|^2 + |\overrightarrow{BC}|^2 + |\overrightarrow{CA}|^2 \ , \end{aligned}$$

as desired.

120. Determine all pairs of nonnull vectors $\mathbf{x}$, $\mathbf{y}$ for which the following sequence $\{a_n : n = 1, 2, \cdots\}$ is (a) increasing, (b) decreasing, where

$$a_n = |\mathbf{x} - n\mathbf{y}| \ .$$

*Solution 1.* By the triangle inequality, we obtain that

$$|\mathbf{x} - n\mathbf{y}| + |\mathbf{x} - (n-2)\mathbf{y}| \geq 2|\mathbf{x} - (n-1)\mathbf{y}| \ ,$$

whence

$$|\mathbf{x} - n\mathbf{y}| - |\mathbf{x} - (n-1)\mathbf{y}| \geq |\mathbf{x} - (n-1)\mathbf{y}| - |\mathbf{x} - (n-2)\mathbf{y}| \ ,$$

for $n \geq 3$. This establishes that the sequence is never decreasing, and will increase if and only if

$$|\mathbf{x} - 2\mathbf{y}| \geq |\mathbf{x} - \mathbf{y}| \geq 0 \ .$$

This condition is equivalent to

$$\mathbf{x} \cdot \mathbf{x} - 4\mathbf{x} \cdot \mathbf{y} + 4\mathbf{y} \cdot \mathbf{y} \geq \mathbf{x} \cdot \mathbf{x} - 2\mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{y}$$

or $3|\mathbf{y}|^2 \geq 2\mathbf{x} \cdot \mathbf{y}$.

*Solution 2.*

$$\begin{aligned} a_n^2 &= |\mathbf{x}|^2 - 2n(\mathbf{x} \cdot \mathbf{y}) + n^2|\mathbf{y}|^2 \\ &= |\mathbf{y}|^2\left[\left(n - \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{y}|^2}\right)^2\right] + \left[|\mathbf{x}|^2 - \frac{(\mathbf{x} \cdot \mathbf{y})^2}{|\mathbf{y}|^4}\right] \ . \end{aligned}$$

This is a quadratic whose nonconstant part involves the form $(n - c)^2$. This is an increasing function of $n$, for positive integers $n$, if and only if $c \leq 3/2$. Hence, the sequence increases if and only if

$$\frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{y}|^2} \leq \frac{3}{2} \ .$$