PROBLEMS FOR JULY

Please send your solution to
Edward J. Barbeau
Department of Mathematics
University of Toronto
Toronto, ON M5S 3G3

no later than September 10, 2003. It is important that your complete mailing address and your email address appear on the front page.

Notes. A *partition* of the positive integer n is a representation (up to order) of n as a sum of not necessarily distinct positive integers, *i.e.*, $n = a_1 + a_2 + \cdots + a_k$ with $a_1 \geq a_2 \geq \cdots \geq a_k \geq 1$. The number of distinct partitions is denoted by $p(n)$. Thus, $p(6) = 11$ since 6 can be written as $6 = 5+1 = 4+2 = 4+1+1 = 3+3 = 3+2+1 = 3+1+1+1 = 2+2+2 = 2+2+1+1 = 2+1+1+1+1 = 1+1+1+1+1+1$.

241. Determine $\sec 40° + \sec 80° + \sec 169°$.

242. Let ABC be a triangle with sides of length a, b, c oppposite respective angles A, B, C. What is the radius of the circle that passes through the points A, B and the incentre of triangle ABC when angle C is equal to (a) 90°; (b) 120°; (c) 60°. (With thanks to Jean Turgeon, Université de Montréal.)

243. The inscribed circle, with centre I, of the triangle ABC touches the sides BC, CA and AB at the respective points D, E and F. The line through A parallel to BC meets DE and DF produced at the respective points M and N. The modpoints of DM and DN are P and Q respectively. Prove that A, E, F, I, P, Q lie on a common circle.

244. Let $x_0 = 4$, $x_1 = x_2 = 0$, $x_3 = 3$, and, for $n \geq 4$, $x_{n+4} = x_{n+1} + x_n$. Prove that, for each prime p, x_p is a multiple of p.

245. Determine all pairs (m, n) of positive integers with $m \leq n$ for which an $m \times n$ rectangle can be tiles with congrent pieces formed by removing a 1×1 square from a 2×2 square.

246. Let $p(n)$ be the number of partitions of the positive integer n, and let $q(n)$ denote the number of finite sets $\{u_1, u_2, u_3, \cdots, u_k\}$ of positive integers that satisfy $u_1 > u_2 > u_3 > \cdots > u_k$ such that $n = u_1 + u_3 + u_5 + \cdots$ (the sum of the ones with odd indices). Prove that $p(n) = q(n)$ for each positive integer n.

    For example, $q(6)$ counts the sets $\{6\}$, $\{6, 5\}$, $\{6, 4\}$, $\{6, 3\}$, $\{6.2\}$, $\{6, 1\}$, $\{5, 4, 1\}$, $\{5, 3, 1\}$, $\{5, 2, 1\}$, $\{4, 3, 2\}$, $\{4, 3, 2, 1\}$.

247. Let $ABCD$ be a convex quadrilateral with no pairs of parallel sides. Associate to side AB a point T as follows. Draw lines through A and B parallel to the opposite side CD. Let these lines meet CB produced at B' and DA produced at A', and let T be the intersection of AB and $B'A'$. Let U, V, W be points similarly constructed with respect to sides BC, CD, DA, respectively. Prove that $TUVW$ is a parallelogram.