PROBLEMS FOR AUGUST

Please send your solutions to
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no later than October 15, 2004. It is important that your complete mailing address and your email address appear on the front page. If you do not write your family name last, please underline it.

324. (*Correction.*) The base of a pyramid $ABCDV$ is a rectangle $ABCD$ with $|AB| = a$, $|BC| = b$ and $|VA| = |VB| = |VC| = |VD| = c$. Determine the area of the intersection of the pyramid and the plane parallel to the base VA that contains the diagonal BD.

325. Solve for positive real values of x, y, t:
$$(x^2 + y^2)^2 + 2tx(x^2 + y^2) = t^2y^2 \ .$$
Are there infinitely many solutions for which the values of x, y, t are all positive integers?

Optional rider: What is the smallest value of t for a positive integer solution?

326. In the triangle ABC with semiperimeter $s = \frac{1}{2}(a+b+c)$, points U, V, W lie on the respective sides BC, CA, AB. Prove that
$$s < |AU| + |BV| + |CW| < 3s \ .$$
Give an example for which trhe sum in the middle is equal to $2s$.

327. Let A be a point on a circle with centre O and let B be the midpoint of OA. Let C and D be points on the circle on the same side of OA produced for which $\angle CBO = \angle DBA$. Let E be the midpoint of CD and let F be the point on EB produced for which $BF = BE$.

(a) Prove that F lies on the circle.

(b) What is the range of angle EAO?

328. Let $\mathcal{C}$ be a circle with diameter AC and centre D. Suppose that B is a point on the circle for which $BD \perp AC$. Let E be the midpoint of DC and let Z be a point on the radius AD for which $EZ = EB$.

Prove that

(a) The length c of BZ is the length of the side of a regular pentagon inscribed in $\mathcal{C}$.

(b) The length b of DZ is the length of the side of a regular decagon (10-gon) inscribed in $\mathcal{C}$.

(c) $c^2 = a^2 + b^2$ where a is the length of a regular hexagon inscribed in $\mathcal{C}$.

(d) $(a + b) : a = a : b$.

329. Let x, y, z be positive real numbers. Prove that
$$\sqrt{x^2 - xy + y^2} + \sqrt{y^2 - yz + z^2} \geq \sqrt{x^2 + xz + z^2} \ .$$

330. At an international conference, there are four official languages. Any two participants can communicate in at least one of these languages. Show that at least one of the languages is spoken by at least 60% of the participants.

331. Some checkers are placed on various squares of a $2m \times 2n$ chessboard, where m and n are odd. Any number (including zero) of checkers are placed on each square. There are an odd number of checkers in each row and in each column. Suppose that the chessboard squares are coloured alternately black and white (as usual). Prove that there are an even number of checkers on the black squares.