Problems for JANUARY

Please send your solution to
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no later than February 29, 2004. It is important that your complete mailing address and your email address appear on the front page.

283. (a) Determine all quadruples (a, b, c, d) of positive integers for which the greatest common divisor of its elements is 1,

$$\frac{a}{b} = \frac{c}{d}$$

and $a + b + c = d$.

(b) Of those quadruples found in (a), which also satisfy

$$\frac{1}{b} + \frac{1}{c} + \frac{1}{d} = \frac{1}{a} \ ?$$

(c) For quadruples (a, b, c, d) of positive integers, do the conditions $a+b+c = d$ and $(1/b)+(1/c)+(1/d) = (1/a)$ together imply that $a/b = c/d$?

284. Suppose that $ABCDEF$ is a convex hexagon for which $\angle A + \angle C + \angle E = 360°$ and

$$\frac{AB}{BC} \cdot \frac{CD}{DE} \cdot \frac{EF}{FA} = 1 \ .$$

Prove that

$$\frac{AB}{BF} \cdot \frac{FD}{DE} \cdot \frac{EC}{CA} = 1 \ .$$

285. (a) Solve the following system of equations:

$$(1 + 4^{2x-y})(5^{1-2x+y}) = 1 + 2^{2x-y+1} \ ;$$
$$y^2 + 4x = \log_2(y^2 + 2x + 1) \ .$$

(b) Solve for real values of x:

$$3^x \cdot 8^{x/(x+2)} = 6 \ .$$

Express your answers in a simple form.

286. Construct inside a triangle ABC a point P such that, if X, Y, Z are the respective feet of the perpendiculars from P to BC, CA, AB, then P is the centroid (intersection of the medians) of triangle XYZ.

287. Let M and N be the respective midpoints of the sides BC and AC of the triangle ABC. Prove that the centroid of the triangle ABC lies on the circumscribed circle of the triangle CMN if and only if

$$4 \cdot |AM| \cdot |BN| = 3 \cdot |AC| \cdot |BC| \ .$$

288. Suppose that $a_1 < a_2 < \cdots < a_n$. Prove that

$$a_1a_2^4 + a_2a_3^4 + \cdots + a_na_1^4 \geq a_2a_1^4 + a_3a_2^4 + \cdots + a_1a_n^4 \ .$$

289. Let $n(r)$ be the number of points with integer coordinates on the circumference of a circle of radius $r > 1$ in the cartesian plane. Prove that

$$n(r) < 6\sqrt[3]{\pi r^2} \ .$$