PROBLEMS FOR APRIL, 2005

Please send your solution to
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no later than April 30, 2005 It is important that your complete mailing address and your email address appear on the front page. If you do not write your family name last, please underline it.

374. What is the maximum number of numbers that can be selected from $\{1, 2, 3, \cdots, 2005\}$ such that the difference between any pair of them is not equal to 5?

375. Prove or disprove: there is a set of concentric circles in the plane for which both of the following hold:

 (i) each point with integer coordinates lies on one of the circles;

 (ii) no two points with integer coefficients lie on the same circle.

376. A soldier has to find whether there are mines buried within or on the boundary of a region in the shape of an equilateral triangle. The effective range of his detector is one half of the height of the triangle. If he starts at a vertex, explain how he can select the shortest path for checking that the region is clear of mines.

377. Each side of an equilateral triangle is divided into 7 equal parts. Lines through the division points parallel to the sides divide the triangle into 49 smaller equilateral triangles whose vertices consist of a set of 36 points. These 36 points are assigned numbers satisfying both the following conditions:

 (a) the number at the vertices of the original triangle are 9, 36 and 121;

 (b) for each rhombus composed of two small adjacent triangles, the sum of the numbers placed on one pair of opposite vertices is equal to the sum of the numbers placed on the other pair of opposite vertices.

 Determine the sum of all the numbers. Is such a choice of numbers in fact possible?

378. Let $f(x)$ be a nonconstant polynomial that takes only integer values when x is an integer, and let P be the set of all primes that divide $f(m)$ for at least one integer m. Prove that P is an infinite set.

379. Let n be a positive integer exceeding 1. Prove that, if a graph with $2n + 1$ vertices has at least $3n + 1$ edges, then the graph contains a circuit (*i.e.*, a closed non-self-intersecting chain of edges whose terminal point is its initial point) with an even number of edges. Prove that this statement does not hold if the number of edges is only $3n$.

380. Factor each of the following polynomials as a product of polynomials of lower degree with integer coefficients:

 (a) $(x + y + z)^4 - (y + z)^4 - (z + x)^4 - (x + y)^4 + x^4 + y^4 + z^4$;

 (b) $x^2(y^3 - z^3) + y^2(z^3 - x^3) + z^2(x^3 - y^3)$;

 (c) $x^4 + y^4 - z^4 - 2x^2y^2 + 4xyz^2$;

 (d) $(yz + zx + xy)^3 - y^3z^3 - z^3x^3 - x^3y^3$;

 (e) $x^3y^3 + y^3z^3 + z^3x^3 - x^4yz - xy^4z - xyz^4$;

 (f) $2(x^4 + y^4 + z^4 + w^4) - (x^2 + y^2 + z^2 + w^2)^2 + 8xyzw$;

 (g) $6(x^5 + y^5 + z^5) - 5(x^2 + y^2 + z^2)(x^3 + y^3 + z^3)$.