PROBLEMS FOR FEBRUARY, 2005

Please send your solution to
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no later than February 28, 2005 It is important that your complete mailing address and your email address appear on the front page. If you do not write your family name last, please underline it.

348. (b) Syppose that $f(x)$ is a real-valued function defined for real values of x. Suppose that both $f(x) - 3x$ and $f(x) - x^3$ are increasing functions. Must $f(x) - x - x^2$ also be increasing on all of the real numbers, or on at least the positive reals?

360. Eliminate θ from the two equations

$$x = \cot\theta + \tan\theta$$

$$y = \sec\theta - \cos\theta \ ,$$

to get a polynomial equation satisfied by x and y.

361. Let $ABCD$ be a square, M a point on the side BC, and N a point on the side CD for which $BM = CN$. Suppose that AM and AN intersect BD and P and Q respectively. Prove that a triangle can be constructed with sides of length $|BP|$, $|PQ|$, $|QD|$, one of whose angles is equal to 60°.

362. The triangle ABC is inscribed in a circle. The interior bisectors of the angles A, B, C meet the circle again at U, V, W, respectively. Prove that the area of triangle UVW is not less than the area of triangle ABC.

363. Suppose that x and y are positive real numbers. Find all real solutions of the equation

$$\frac{2xy}{x+y} + \sqrt{\frac{x^2+y^2}{2}} = \sqrt{xy} + \frac{x+y}{2} \ .$$

364. Determine necessary and sufficient conditions on the positive integers a and b such that the vulgar fraction a/b has the following property: *Suppose that one successively tosses a coin and finds at one time, the fraction of heads is less than a/b and that at a later time, the fraction of heads is greater than a/b; then at some intermediate time, the fraction of heads must be exactly a/b.*

365. Let $p(z)$ be a polynomial of degree greater than 4 with complex coefficients. Prove that $p(z)$ must have a pair u, v of roots, not necessarily distinct, for which the real parts of both u/v and v/u are positive. Show that this does not necessarily hold for polynomials of degree 4.

366. What is the largest real number r for which

$$\frac{x^2+y^2+z^2+xy+yz+zx}{\sqrt{x}+\sqrt{y}+\sqrt{z}} \geq r$$

holds for all positive real values of x, y, z for which $xyz = 1$.