PROBLEMS FOR MARCH, 2005

Please send your solution to
Prof. Edward J. Barbeau
Department of Mathematics
University of Toronto
Toronto, ON M5S 3G3

no later than April 15, 2005 It is important that your complete mailing address and your email address appear on the front page. If you do not write your family name last, please underline it.

367. Let a and c be fixed real numbers satisfying $a \leq 1 \leq c$. Determine the largest value of b that is consistent with the condition

$$a + bc \leq b + ac \leq c + ab \ .$$

368. Let A, B, C be three distinct points of the plane for which $AB = AC$. Describe the locus of the point P for which $\angle APB = \angle APC$.

369. $ABCD$ is a rectangle and APQ is an inscribed equilateral triangle for which P lies on BC and Q lies on CD.

(a) For which rectangles is the configuration possible?

(b) Prove that, when the configuration is possible, then the area of triangle CPQ is equal to the sum of the areas of the triangles ABP and ADQ.

370. A deck of cards has nk cards, n cards of each of the colours C_1, C_2, $\cdots$, C_k. The deck is thoroughly shuffled and dealt into k piles of n cards each, P_1, P_2, $\cdots$, P_k. A game of solitaire proceeds as follows: The top card is drawn from pile P_1. If it has colour C_i, it is discarded and the top card is drawn from pile P_i. If it has colour C_j, it is discarded and the top card is drawn from pile P_j. The game continues in this way, and will terminate when the nth card of colour C_1 is drawn and discarded, as at this point, there are no further cards left in pile P_1. What is the probability that every card is discarded when the game terminates?

371. Let X be a point on the side BC of triangle ABC and Y the point where the line AX meets the circumcircle of triangle ABC. Prove or disprove: if the length of XY is maximum, then AX lies between the median from A and the bisector of angle BAC.

372. Let b_n be the number of integers whose digits are all $1, 3, 4$ and whose digits sum to n. Prove that b_n is a perfect square when n is even.

373. For each positive integer n, define

$$a_n = 1 + 2^2 + 3^3 + \cdots + n^n \ .$$

Prove that there are infinitely many values of n for which a_n is an odd composite number.