Solutions to the August problems

395. None of the nine participants at a meeting speaks more than three languages. Two of any three speakers speak a common language. Show that there is a language spoken by at least three participants.

Solution 1. `Case (i).` *Each pair has a common language.* There are $\binom{9}{2} = 36$ pairs with at most $3 \times 9 = 27$ languages. By the Pigeonhole Principle, there exists a language spoken by at least two different pairs, which includes either three or four participants.

`Case (ii).` *There exists a pair consisting of, say, A and B, who have no language in common.* These two know k languages between them, where $2 \le k \le 6$. Let the other participants be $P_1, \cdots, P_7$. Consider each triplet $\{A, B, P_i\}$, we see that each participant P_i knows at least one of the k languages. By the Pigeonhole Principle, one of the k languages is known by at least two of the P_i along with either A or B. The result follows.

Solution 2. [F. Barekat] If every speaker speaks a language shared by two other participants, then the result holds. On the other hand, suppose that there is a participant P each of whose languages is shared by at most one other person. Then at most three people speak one of P's languages and there are five participants, A, B, C, D, E, who have no language in common with P. Considering each pair of these five with P, we see that each pair of the five participants speaks a common language. Thus, A shares a language with each of B, C, D, E, and, speaking at most three languages, must share the same language with two of them. The result follows.

Solution 3. [A. Kong] We prove the result by contradiction. Suppose if possible that no language is spoken by more than two participants. Form a graph with 9 vertices corresponding to the participants and connect two vertices by an edge if and only if the corresponding participants have a common language. The number of edges does not exceed the number of languages. There are $\binom{9}{3} = 84$ triplets of vertices, each of which involves at least one edge. Any edge can be involved in at most 7 triplets. As two edges can be involved in 14 distinct triplets only if the edges have no common vertex, we can find at most four edges that can be involved in seven triplets each with no two triplets having an edge in common. Since each vertex can be an endpoint of at most three edges and each edge involves two vertices, the number of edges is at most $\lfloor \frac{1}{2}(3 \times 9) \rfloor = 13 = 7 + 6$. Hence, the number of triplets that are involved with these edges is at most $4 \times 7 + 9 \times 6 = 82$, a contradiction.

Comment. The hypothesis allows for the possibility that some participant may know *fewer* that three languages, so you should not base your argument on everyone knowing exactly three languages. This is a situation where a contradiction argument can be avoided, and you should try to do so.

396. Place 32 white and 32 black checkers on a 8×8 square chessboard. Two checkers of different colours form a *related pair* if they are placed in either the same row or the same column. Determine the maximum and the minimum number of related pairs over all possible arrangements of the 64 checkers.

Solution. The maximum number of related pairs is 256, achieved by putting 4×4 blocks of black checkers in diagonally opposite corners of the board and 4×4 blocks of white checkers in the other diagonally opposite corner, or else by alternating the colours as on a standard chessboard. The minimum number of related pairs is 128, achieved by filling four columns entirely with black checkers and the other four columns with white checkers. We now prove that these bounds hold.

Suppose that in a given row or column, there are x black and $8 - x$ white checkers. Then the number of related pairs is

$$x(8 - x) = 16 - (x - 4)^2 \le 16$$

with equality if and only if $x = 4$. Hence the total number of related pairs cannot exceed $16 \times 16 = 256$.

The number of related pairs is independent of the order in which the rows or columns of checkers appear, so, wolog, we may suppose that the number of white checkers in the columns decreases as we go from left to right. Suppose that there is a white checker to the right of a black checker in some row, so that the white checker appears in a column with r white checkers and the black checker appears in a column with s white checkers, where $r \leq s$. Suppose now that we interchange the positions of just these two checkers. The number of related pairs in the rows remains unchanged, while the number of related pairs in the columns gets reduced by

$$[r(8-r)+s(8-s)]-[(r-1)(9-r)+(s+1)(7-s)]=2(s+1-r)>0 \ .$$

We can continue this sort of exchange, reducing the number of related pairs each time, until every white checker is to the left of all the black checkers in the same row. Thus, we need consider only configurations in which no black checker is to the left of or above white checker.

Suppose that the ith row has x_i white checkers and the jth column has y_j white checkers, where $8 \geq x_1 \geq x_2 \geq \cdots \geq x_8 \geq 0$ and $8 \geq y_1 \geq y_2 \geq \cdots \geq y_8 \geq 0$. We have that $a_1 + \cdots + x_8 = y_1 + \cdots + y_8 = 32$. Setting $x_9 = 0$, we see that for each i, $x_i - x_{i+1}$ is equal to the number of indices j for which $y_j = i$.

The total number of related pairs is

$$\begin{aligned}
\sum_{i=1}^{8} x_i(8-x_i) + \sum_{j=1}^{8} y_j(8-y_j) &= 8 \times \sum x_i + 8 \times \sum y_j - \sum x_i^2 - \sum y_j^2 \\
&= 8 \times 32 + 8 \times 32 - (x_1^2 + \cdots + x_8^2) - \sum_{i=1}^{8} (x_i - x_{i+1}) i^2 \\
&= 512 - [(x_1^2 + \cdots + x_8^2) + (x_1 - x_2) + 4(x_2 - x_3) + 9(x_3 - x_4) + \cdots] \\
&= 512 - [(x_1 + x_2^2 + 2x_2 + x_3^2 + 4x_3 + \cdots) + (x_1 + \cdots + x_8)] \\
&= 512 - [x_1^2 + (x_2+1)^2 + \cdots + (x_8+7)^2 - 140 + 32] \\
&= 620 - [x_1^2 + (x_2+1)^2 + \cdots + (x_8+7)^2] \ .
\end{aligned}$$

Thus, we require the maximum of $x_1^2 + (x_2+1)^2 + \cdots + (x_8+7)^2$ when $x_1 + \cdots + x_8 = 32$ and $8 \geq x_1 \geq \cdots \geq x_8 \geq 0$.

At this point, the argument becomes tedious and a simpler one is sought. Let $z_i = x_i + (i-1)$ with $1 \leq i \leq 8$. It is straightforward to check that $x_5 \leq 6$, $x_6 \leq 5$, $x_7 \leq 4$ and $x_8 \leq 4$, so that $z_1 \leq 8$, $z_2 \leq 9$, $z_3 \leq 10$, $z_4 \leq 11$, $z_5 \leq 10$, $z_6 \leq 10$, $z_7 \leq 10$ and $z_8 \leq 11$. The value 11 is possible for z_i only when $i = 4$ and we must have $(x_1, \cdots, x_8) = (8,8,8,8,0,0,0,0)$ or $i = 8$ and we must have $(x_1, \cdots, x_8) = (4,4,4,4,4,4,4,4)$. In both cases, the square sum is 492. In a similar way, we find that $x_1 \geq 4$, $x_2 \geq 4$, $x_3 \geq 3$, $x_4 \geq 2$ and so $z_1 \geq 4$, $z_2 \geq 5$, $z_3 \geq 5$, $z_4 \geq 5$, $z_5 \geq 4$, $z_6 \geq 5$, $z_7 \geq 6$, $z_8 \geq 7$. The value 4 is possible for z_i only when $i = 1$ or $i = 5$ and we have $(x_1, \cdots, x_8) = (8,8,8,8,0,0,0,0)$ or $i = 8$ and we must have $(x_1, \cdots, x_8) = (4,4,4,4,4,4,4,4)$. Otherwise, we must have $5 \leq z_i \leq 10$ for each i, and checking out the possibilities leads to square sums less than 492.

397. The altitude from A of triangle ABC intersects BC in D. A circle touches BC at D, intersects AB at M and N, and intersects AC at P and Q. Prove that

$$(AM + AN) : AC = (AP + AQ) : AB \ .$$

Solution 1. Let the circle intersect AD again at E.

$$\begin{aligned}
(AE + AD) \cdot AD = AE \cdot AD + AD^2 &= AM \cdot AN + AB^2 - BD^2 \\
&= AM \cdot AN + AB^2 - BN \cdot BM = AM \cdot AN + (AN + NB) \cdot (AM + MB) - BN \cdot BM \\
&= AM \cdot AN + AN \cdot AB + NB \cdot AM = AM \cdot (AN + NB) + AN \cdot AB \\
&= (AM + AN) \cdot AB \ .
\end{aligned}$$

Similarly, $(AE + AD) \cdot AD = (AP + AQ) \cdot AC$. The result follows.

Solution 2. [F. Barekat] Let O be the centre of the circle, and let S and T be the respective midpoints of MN and PQ. Then $OS \perp AB$, $OT \perp AC$,

$$AM + AN = 2AS = 2AO\cos\angle BAD = 2AO\sin\angle ABC$$

and

$$AP + AQ = 2AT = 2AO\cos\angle CAD = 2AO\sin\angle ACB\ .$$

Hence

$$AB : AC = \sin\angle ACB : \sin\angle ABC = (AP + AQ) : (AM + AN)$$

as desired.

398. Given three disjoint circles in the plane, construct a point in the plane so that all three circles subtend the same angle at that point.

Solution. If two circles of radii r and R are given with respective centres O and P, and if Q is a point at which both circles subtend equal angles, then $OQ : OP = r : R$. To prove this, draw tangents from Q to meet the circles of centres O and P at A and B respectively, so that $\angle AQO$ and $\angle BQP$ are half the subtended angles. Then the proportion is a consequence of the similarity of the triangles QAO and QBP.

Suppose first that $r < R$. The locus of Q turns out to be a circle (a circle of Apollonius). One way to see this is to introduce coordinates with O at the origin, P at $(p, 0)$ and Q at (x, y). The equation of the locus is $\sqrt{x^2 + y^2} = (r/R)\sqrt{(x - p)^2 + y^2}$. This simplifies to $(R^2 - r^2)(x^2 + y^2) + 2pr^2x - p^2r^2 = 0$, the equation of a circle. Let one pair of common tangents to the circles intersect at the point V on the same side of both circles and the other pair at W between the two circles. V is the centre of a dilation with factor r/R that takes the larger circle to the smaller, and W is the centre of a dilation with factor $-r/R$ that takes the larger circle to the smaller. V and W both lie on the locus and form a line of symmetry for the locus; hence it is a diameter of the locus circle. So to construct the locus, it suffices to determine the points V and W. This can be done for example by drawing parallel diameters to the two circles and noting that the line joining pairs of their endpoints must pass through either V or W (since the dilations takes one diameter to the other).

To solve the problem, for each of two pairs of the three circles, determine circle at which the two circles subtend equal angles. If these circles intersect, then the intersection points will be points at which all three circles subtend equal angles.

It remains to consider the case where at least two of the circles have the same radius. In this case, a reflection about the right bisector of the line of centres takes one circle to the other, and this right bisector is the locus desired. So the desired point is in this case, either the intersection of a line and a circle or of two lines.

399. Let n and k be positive integers for which $k < n$. Determine the number of ways of choosing k numbers from $\{1, 2, \cdots, n\}$ so that no three consecutive numbers appear in any choice.

Solution 1. An admissible choice of k numbers corresponds to a sequence of k 1's and $n - k$ 0's in a sequence of n terms where 1 appears in the ith position if and only if i is selected as one of the k numbers and three 1's do not appear consecutively. We count the number of such sequences.

Suppose that an admissible sequence has a occurrences of 11 and b occurrences of 1, separated by 0's, so that $k = 2a+b$. The patterns of two 1's can be interpolated among the patterns of one 1 in $\binom{a+b}{a}$ ways. $a+b-1$ zeros must be placed in that many slots between adjacent patterns of 11 and 1. (If $a + b - 1 > n - k$, then a sequence of the type specified cannot occur, but this will automatically come out in the final expression.)

If $a + b - 1 \leq n - k$, then $(n - k) - (a + b - 1)$ 0's remain to be allocated, either at the beginning or the end of the sequence or in the $a + b - 1$ slots that already contain one 0. Recall that u identical objects can be distributed among v distinguishable boxes in $\binom{u+v-1}{v-1}$ ways. (To see this, place $u + v - 1$ identical objects in a line; select $v-1$ gaps between adjacent pairs to determine a partition into v boxes each with at least one object; now, remove the superfluous v objects, one from each box.) Applying this to the present situation, we deduce that the spare $(n-k)-(a+b-1)$ 0's can be distributed in

$$\binom{n-k-(a+b-1)+(a+b-1)-1}{a+b-1} = \binom{n-k-1}{k-a}$$

ways.

Thus, the total number of ways of selecting an admissible set of k numbers form $\{1,2,\cdots,n\}$ is

$$\sum\{\binom{a+b}{b}\binom{n-k+1}{a+b} : 2a+b=k, a\geq 0, b\geq 0\} = \sum_{a\geq 0}\binom{k-a}{a}\binom{n-k+1}{k-a} .$$

Solution 2. [K. Kim] Reformulate the problem as selecting a sequence of n integers with k ones and $n-k$ zeros, with 1 being selected if and only if i is selected from among $\{1,2,\cdots,n\}$. At most two ones can appear side by side. Begin with the $n-k$ zeros; there are $n-k+1$ "slots" separated by the zeros into which we may insert 0, 1 or 2 ones. Suppose that we have i pairs of ones in the slots, where $0\leq 2i\leq k$. We can pick the slots for these in $\binom{n-k+1}{i}$ ways. There are $n-k+1-i$ slots left over and we can fit the remaining singelton ones in them in $\binom{n-k+1-i}{k-2i}$ ways. Hence the total number of ways is

$$\sum\left\{\binom{n-k+1}{i}\binom{n-k+1-i}{k-2i} : 0\leq 2i\leq k\right\} .$$

Solution 3. [F. Barekat] Recall that

$$(1-x)^{-t} = \sum_{i=0}^{\infty}\binom{t+i-1}{t-1}x^i .$$

We transform the given problem into an equivalent problem. Each choice $(b_1,\cdots,b_k)$ of k distinct numbers from $\{1,2,\cdots,m\}$ given in increasing order corresponds to a choice $(c_1,\cdots,c_k)$ of k not necessarily distinct numbers from $\{1,2,\cdots,n-k+1\}$ given in increasing order with $c_i=b_i-(i-1)$ $(1\leq i\leq k)$.

Three of the numbers b_i are consecutive if and only if the corresponding three numbers c_i are equal. Hence the answer to the problem is equal to the number of choices of k numbers from $\{1,2,\cdots,n-k+1\}$ for which at most two numbers are equal to any value.

For each i with $1\leq i\leq n-k+1$, construct the quadratic $1+x+x^2$ and define the generating function

$$\begin{aligned}
f(x) &= (1+x+x^2)^{n-k+1} = (1-x^3)^{n-k+1}(1-x)^{-(n-k+1)} \\
&= \sum_{j=0}^{\infty}(-1)^j\binom{n-k+1}{j}x^{3j}\sum_{i=0}^{\infty}\binom{n+i-k}{n-k}x^i \\
&= \sum_{k=0}^{\infty}\left[\sum_{j=0}^{\infty}(-1)^j\binom{n-k+1}{j}\binom{n-3j}{n-k}\right]x^k .
\end{aligned}$$

For each k, the coefficient counts the number of ways we can form x^k in the expression $f(x)$ by selecting $1, x$ or x^2 from the ith factor. This is the answer to the problem.

Comment. A similar argument to that of the third solution gives $\sum_{j=0}^{\infty}(-1)^j\binom{n-k+1}{j}\binom{n-rj}{n-k}$ for the number of choices that avoid r consecutive integers.

If $f(n,k)$ represents the number of admissible selections, then, for $n\geq 1$, $f(n,1)=n$ and $f(n,2)=\binom{n}{2}$ and for $n\geq 3$, $f(n,3)=\binom{n}{3}-(n-2)=\frac{1}{6}(n-2)(n-3)(n+2)$.

When $n \geq 5$. $k \geq 4$, we can develop some recursion relations. An admissible set of k numbers can be selected from $\{1, 2, \cdots, n\}$ not including n in $f(n-1, k)$ ways. Or we can select n and the remaining $k-1$ numbers from $\{1, 2, \cdots, n-1\}$ provided that both $n-2$ and $n-1$ are not selected in $f(n-1, k-1) - f(n-4, k-3)$ ways. Thus,

$$f(n,k) = f(n-1,k) + f(n-1,k-1) - f(n-4,k-3) \ .$$

Alternatively, we can look at the three cases where n is not chosen, where n is chosen but $n-1$ is not and where both n and $n-1$ are chosen but $n-2$ is not. This yields that

$$f(n,k) = f(n-1,k) + f(n-2,k-1) + f(n-3,k-2) \ .$$

In particular, we have that $f(n,4) = \binom{n}{4} - (n-3)^2 = \frac{1}{24}(n-3)(n-4)(n^2+n-18)$ and $f(n,5) = \binom{n}{5} - \frac{1}{2}(n-3)(n-4)^2 = \frac{1}{120}(n-3)(n-4)(n-5)(n-6)(n+8)$.

We have the following table of values

n ↓	$k \rightarrow$	1	2	3	4	5	6	7
1		1						
2		2	1					
3		3	3	0				
4		4	6	2	0			
5		5	10	7	1	0		
6		6	15	16	6	0	0	
7		7	21	30	19	3	0	0
8		8	28	50	45	16	1	0
9		9	36	77	90	51	10	0

400. Let a_r and b_r $(1 \leq r \leq n)$ be real numbers for which $a_1 \geq a_2 \geq \cdots \geq a_n > 0$ and

$$b_1 \geq a_1 \ , \quad b_1 b_2 \geq a_1 a_2 \ , \quad b_1 b_2 b_3 \geq a_1 a_2 a_3 \ , \quad \cdots \quad , b_1 b_2 \cdots b_n \geq a_1 a_2 \cdots a_n \ .$$

Show that

$$b_1 + b_2 + \cdots + b_n \geq a_1 + a_2 + \cdots + a_n \ .$$

Solution. Since $b_1 \cdots b_s > 0$ for all s, each b_i is positive. Let $c_0 = 1$ and define

$$c_1 = \frac{b_1}{a_1} \ , \quad c_2 = \frac{b_1 b_2}{a_1 a_2} \ , \quad \cdots, \quad c_n = \frac{b_1 b_2 \cdots b_n}{a_1 a_2 \cdots a_n} \ .$$

Then $c_i \geq 1$ and

$$b_i = \frac{c_i}{c_{i-1}} a_i$$

for $1 \leq i \leq n$. We have that

$$\begin{aligned}
&(b_1 + \cdots + b_n) - (a_1 + \cdots + a_n) \\
&\qquad = \left(\frac{c_1}{c_0} - 1\right) a_1 + \left(\frac{c_2}{c_1} - 1\right) a_2 + \cdots + \left(\frac{c_n}{c_{n-1}} - 1\right) a_n \\
&\qquad = (c_1 - 1)(a_1 - a_2) + \left(c_1 + \frac{c_2}{c_1} - 2\right)(a_2 - a_3) + \left(c_1 + \frac{c_2}{c_1} + \frac{c_3}{c_2} - 3\right)(a_3 - a_4) + \cdots \\
&\qquad\quad + \left(c_1 + \frac{c_2}{c_1} + \cdots + \frac{c_i}{c_{i-1}} - i\right)(a_i - a_{i-1}) + \cdots + \left(c_1 + \frac{c_2}{c_1} + \cdots + \frac{c_n}{c_{n-1}} - n\right) a_n \ .
\end{aligned}$$

By the Arithmetic-Geometric Means Inequality,

$$\frac{1}{i}\left[c_1 + \frac{c_2}{c_1} + \cdots + \frac{c_i}{c_{i-1}}\right] \geq \left[c_1\left(\frac{c_2}{c_1}\right)\cdots\left(\frac{c_i}{c_{i-1}}\right)\right]^{1/i} = c_i^{1/i} \geq 1 \ ,$$

so that $c_1 + (c_2/c_1) + \cdots + (c_i/c_{i-1}) \geq i$ and the result follows.

Comments. The transformation of the series $\sum((c_i/c_{i-1} - 1)a_i$ is a standard way of dealing with series, known as *summation by parts* and analogous to the calculus technique of integration by parts. It is used as a means of incorporating the hypothesis $a_1 \geq a_2 \geq \cdots \geq a_n > 0$.

An interesting argument comes from F. Barekat who claims a stronger result. The proof almost works, but there is a small fly in the ointment. It is not clear to me that this can be worked around or whether the claimed result is false and a counterexample can be found. We suppose that a_n and b_n are defined for all n such that $b_1b_2\cdots b_n \geq a_1a_2\cdots a_n > 0$. It is claimed that, for all n,

$$(b_1 + b_2 + \cdots + b_n) - (a_1 + a_2 + \cdots + a_n) \geq a_n\left(1 - \frac{a_1a_2\cdots a_n}{b_1b_2\cdots b_n}\right) .$$

When $n = 1$, we have that

$$b_1 - a_1 = b_1\left(1 - \frac{a_1}{b_1}\right) \geq a_1\left(1 - \frac{a_1}{b_1}\right) .$$

Suppose that the result holds for $n = m$. Then

$$\begin{aligned}
&(b_1 + \cdots + b_m + b_{m+1}) - (a_1 + \cdots + a_m + a_{m+1}) - a_{m+1}\left(1 - \frac{a_1\cdots a_m a_{m+1}}{b_1\cdots b_m b_{m+1}}\right) \\
&\geq a_m\left(1 - \frac{a_1\cdots a_m}{b_1\cdots b_m}\right) + b_{m+1} + \frac{a_{m+1}a_1\cdots a_{m+1}}{b_1\cdots b_{m+1}} - 2a_{m+1} \\
&= \frac{1}{b_1\cdots b_{m+1}}[a_m b_1\cdots b_m b_{m+1} - a_m b_{m+1} a_1\cdots a_m + b_{m+1}b_1\cdots b_{m+1} + a_{m+1}a_1\cdots a_{m+1} \\
&\qquad - 2a_{m+1}b_1\cdots b_{m+1}] \\
&= \frac{1}{b_1\cdots b_{m+1}}[b_1\cdots b_m(a_m b_{m+1} + b_{m+1}^2 - 2a_{m+1}b_{m+1}) - (a_1\cdots a_m)(a_m b_{m+1} - a_{m+1}^2)] \\
&= \frac{1}{b_1\cdots b_{m+1}}[(b_1\cdots b_m - a_1\cdots a_m)(a_m b_{m+1} + b_{m+1}^2 - 2a_{m+1}b_{m+1}) \\
&\qquad + (a_1\cdots a_m)(b_{m+1}^2 - 2a_{m+1}b_{m+1} + a_{m+1}^2)] \\
&= \frac{1}{b_1\cdots b_{m+1}}[b_{m+1}(b_1\cdots b_m - a_1\cdots a_m)(a_m + b_{m+1} - 2a_{m+1}) + (a_1\cdots a_m)(b_{m+1} - a_{m+1})^2] \\
&= \frac{1}{b_1\cdots b_{m+1}}[b_{m+1}(b_1\cdots b_m - a_1\cdots a_m)((a_m - a_{m+1}) \\
&\qquad + (b_{m+1} - a_{m+1})) + (a_1\cdots a_m)(b_{m+1} - a_{m+1})^2] .
\end{aligned}$$

The argument can be completed if $b_{m+1} \geq a_{m+1}$, but seems to be in trouble otherwise.

401. Five integers are arranged in a circle. The sum of the five integers is positive, but at least one of them is negative. The configuration is changed by the following moves: at any stage, a negative integer is selected and its sign is changed; this negative integer is added to each of its (immediate) neighbours (*i.e.*, its absolute value is subtracted from each of its neighbours).

    Prove that, regardless of the negative number selected for each move, the process will eventually terminate with all integers nonnegative in exactly the same number of moves with exactly the same configuration.

Solution. We associate with each arrangement of numbers a doubly infinite sequence, and analyze how the corresponding sequence alters with each move. Suppose the numbers, given clockwise, are a, b, c, d, e, and that their positive sum is s. Pick one number, say a, as a starting point and construct the sequence of running totals as we proceed closkwise: $a, a+b, a+b+c, a+b+c+d, s, s+a, \cdots$. This produces a sequence of blocks of five numbers for which each block of five is obtained from a previous one by adding s to its entries, Now extend the sequence backwards, preserving this "quasi-periodic" property.

We make some observations. The (doubly-infinite) sequence is increasing if and only if all entries in the circle are nonnegative. If the circle has a negative entry, then the sequence decreases at this particular entry. Given any term in the sequence, there are at most finitely many terms following it that do not exceed it.

Suppose, without loss of generality, that the number c in the circle is negative and that the sum in the sequence up to a is t. Then we have the consecutive terms: t, $t+b$, $t+b+c$, $t+b+c+d$, $\cdots$; note that $t+b+c < t+b$. Now perform the operation on the five numbers, selecting c as the relevant negative number. Then, the numbers in the circle become a, $b+c$, $-c$, $d+c$, e and the corresponding terms in the sequence are t, $t+b+c$, $t+b$, $t+b+c+d$, $\cdots$. In other words, the effect on the sequence is that, for every pair of entries corresponding to b and c, the terms get interchanged, and a decreasing pair become increasing.

We define the following isomorphism (mathematically equivalent situation). Each configuration of five integers of five integers in a circle with a designated starting entry (a) for the sequence corresponds to a doubly infinite sequence that has the quasiperiodicity defined above, and every such sequence gives rise to a circle of five integers. The operation of the problem corresponds to the switching of periodic sets of decreasing pairs to increasing pairs. Note that the entries of the doubly infinite sequence stay the same; they just get rearranged.

Suppose that we focus on five consecutive positions in the original doubly infinite sequence. Each of these has a finite number, say p, of terms following it in the sequence that are smaller, and a finite number, say q, of terms preceding it that are bigger. Each switching operation on two entries will decrease p for one and decrease q for the other. Eventually, each of the five terms in the five original positions will end up $p-q$ positions to the right and we will have an increasing sequence. The result follows.

Comment. We can go part way, showing that the sequence of moves will terminate, by associating with each configuration a positive quantity that decreases. With the integers a, b, c, d, e with $c < 0$ and sum $s > 0$, we form the quantity $(a-c)^2 + (b-d)^2 + (c-e)^2 + (d-a)^2 + (e-b)^2$. If we make a move, pivoting on c, to get $a, b+c, -c, d+c, e$, the corresponding quantity is $(a+c)^2 + (b-d)^2 + (c+e)^2 + (d+c-a)^(e-b-c)^2$. This is smaller than the preceding quantity by the positive amount $(-2c)s$. This difference depends on the size of $|c|$, and so we cannot get a fix on how long it will take to achieve a circle whose integers are all nonnegative.