Solutions to the January problems.

353. The two shortest sides of a right-angled triangle, a and b, satisfy the inequality:

$$\sqrt{a^2 - 6a\sqrt{2} + 19} + \sqrt{b^2 - 4b\sqrt{3} + 16} \leq 3 \ .$$

Find the perimeter of this triangle.

Solution. The equation can be rewritten as

$$\sqrt{(a - 3\sqrt{2})^2 + 1} + \sqrt{(b - 2\sqrt{3})^2 + 4} \leq 3 \ .$$

Since the left side is at least equal to $1 + 2 = 3$, we must have equality and so $a = 3\sqrt{2}$ and $b = 2\sqrt{3}$. The hypotenuse of the triangle equal to $\sqrt{a^2 + b^2} = \sqrt{30}$, and so the perimeter is equal to $3\sqrt{2} + 2\sqrt{3} + \sqrt{30}$. ♠

354. Let ABC be an isosceles triangle with $AC = BC$ for which $|AB| = 4\sqrt{2}$ and the length of the median to one of the other two sides is 5. Calculate the area of this triangle.

Solution. Let M be the midpoint of BC, $\theta = \angle AMB$ and $x = |BM| = |MC|$. Then $|AC| = 2x$. By the Law of Cosines, $4x^2 = x^2 + 25 + 10x\cos\theta$ and $32 = x^2 + 25 - 10x\cos\theta$. Adding these two equations yields that $x^2 = 9$, so that $x = 3$. The height of the triangle from C is $\sqrt{4x^2 - 8} = \sqrt{28} = 2\sqrt{7}$. Hence the area of the triangle is $4\sqrt{14}$. ♠

355. (a) Find all natural numbers k for which $3^k - 1$ is a multiple of 13.

(b) Prove that for any natural number k, $3^k + 1$ is not a multiple of 13.

Solution 1. Let $k = 3q + r$. Since $3^3 \equiv 1 \pmod{13}$, $3^k - 1 \equiv 3^r - 1 \pmod{13}$ and $3^k + 1 \equiv 3^r + 1 \pmod{13}$. Since $3^0 = 1$, $3^2 = 9$, we see that only $3^k - 1$ is a multiple of 13 when k is a multiple of 3. ♠

Solution 2. Let p be a prime and $N = d_0 + d_1 p + \cdots + d_r p^r = (d_r d_{r-1} \cdots d_1 d_0)_p$ be an integer written to base p. Then $p^k = (100 \cdots 00)_p$, $p^k + 1 = (1000 \cdots 01)_p$ and $p^k - 1 = (\overline{p-1}, \overline{p-1}, \cdots \overline{p-1})_p$ where the first two have $k + 1$ digits and the last has k digits. Let $p = 3$, we see that $3^k - 1 = (222 \cdots 22)_3$ and $3^k + 1 = (100 \cdots 01)_3$. Since $13 = (111)_3$, we see that $3^k + 1$ is never a multiple of 13 and $3^k - 1$ is a multiple of 13 if and only if k is a multiple of 3. ♠

356. Let a and b be real parameters. One of the roots of the equation $x^{12} - abx + a^2 = 0$ is greater than 2. Prove that $|b| > 64$.

Solution 1. Clearly, $a \neq 0$. The equation can be rewritten $b = (x^{12} + a^2)/(ax)$. If $x > 2$, then

$$|b| = \frac{x^{12} + a^2}{|a|x} \geq \frac{2|a|x^6}{|a|x} = 2x^5 > 64 \ ,$$

by the arithmetic-geometric means inequality. ♠

Solution 2. [V. Krakovna] The equation can be rewritten

$$x^{12} + \left(\frac{bx}{2} - a\right)^2 = \frac{b^2x^2}{4} \ ,$$

whence $b^2x^2 = 4x^{12} + (bx - 2a)^2 \geq 4x^{12}$ and $b^2 \geq 4x^{10}$. If $|x| > 2$, then $b^2 \geq 2^{12}$ and so $|b| \geq 2^6$. ♠

357. Consider the circumference of a circle as a set of points. Let each of these points be coloured red or blue. Prove that, regardless of the choice of colouring, it is always possible to inscribe in this circle an isosceles triangle whose three vertices are of the same colour.

Solution 1. Consider any regular pentagon inscribed in the given circle. Since there are five vertices and only two options for their colours. the must be three vertices of the same colour. If they are adjacent, then two of the sides of the triangle they determine are sides of the pentagon, and so equal. If two are adjacent and the third opposite to the side formed by the first two, then once again they determine an isosceles triangle. As this covers all the possibilities, the result follows.

Solution 2. If at most finitely many points on the circumference are red, then it is possible to find an isosceles triangle with green vertices. (Why?) Suppose that there are infinitely many red points. Then there are two red points, P and Q, that are neither at the end of a diagonal nor two vertices of an inscribed equilateral triangle. Let U, V, W be three distinct points on the circumference of the circle unequal to P and Q for which $|UP| = |PQ| = |QV|$ and $|PW| = |QW|$. Then the triangles PQU, PQV, PQW and UVW are isoceles. Either one of the first three has red vertices, or the last one has green vertices. ♠

Rider. Can one always find both a red and a green isosceles triangle if there are infinitely many points of each colour?

358. Find all integers x which satisfy the equation

$$\cos\left(\frac{\pi}{8}(3x - \sqrt{9x^2 + 160x + 800})\right) = 1 \ .$$

Solution. We must have that

$$\frac{\pi}{8}(3x - \sqrt{9x^2 + 160x + 800}) = 2k\pi$$

for some integer k, whence

$$3x - \sqrt{9x^2 + 160x + 800} = 16k \ .$$

Multiplying by the surd conjugate of the left side yields

$$-160x - 800 = 16k(3x + \sqrt{9x^2 + 160x + 800})$$

so that

$$3x + \sqrt{9x^2 + 160x + 800} = \frac{1}{k}(-10x - 50) \ .$$

Therefore, $6x = 16k - (1/k)(10x + 50)$, whereupon $(3k + 5)x = 8k^2 - 25$. Multiplying by 9 yields that

$$9x(3k + 5) = 8(9k^2 - 25) - 25 = 8(3k - 5)(3k + 5) - 25 \ ,$$

whereupon $3k + 5$ is a divisor of 25, *i.e.*, one of the six numbers $\pm 1, \pm 5, \pm 25$. This leads to the three possibilities $(k, x) = (-2, -7), (0, -5), (-10, -31)$. The solution $x = -5$ is extraneous, so the given equation has only two integers solutions, $x = -31, -7$. ♠

359. Let ABC be an acute triangle with angle bisectors AA_1 and BB_1, with A_1 and B_1 on BC and AC, respectively. Let J be the intersection of AA_1 and BB_1 (the incentre), H be the orthocentre and O the circumcentre of the triangle ABC. The line OH intersects AC at P and BC at Q. Given that C, A_1, J and B_1 are vertices of a concyclic quadrilateral, prove that $PQ = AP + BQ$.

Solution. [Y. Zhao] Since CA_1JB_1 is concyclic, we have that

$$\angle C = 180^\circ - \angle AJB = \angle JAB + \angle JBA = \frac{1}{2}\angle A + \frac{1}{2}\angle B = 90^\circ - \frac{1}{2}\angle C$$

so that $\angle C = 60°$. (Here we used the fact that the sum of opposite angles of a concyclic quadrilateral is 180° and the sum-of-interior-angles theorem for triangle AJB.) Now, applying the same reasoning to $\angle AHB$ and using the fact that H is the orthocentre of triangle ABC, we find that

$$\begin{aligned}\angle AHB &= 180° - \angle HAB - \angle HBA == 180° - \angle HAB - \angle HCA \\ &= 180° - (90° - \angle ABC) - (90° - \angle BAC) \\ &= \angle ABC + \angle BAC = 180° - \angle ACB = 120° \ .\end{aligned}$$

On the other hand, since O is the circumcentre of triangle ABC, $\angle AOB = 2\angle ACB = 120°$. Therefore, $AOHB$ is concyclic. Now, $\angle PHA + \angle AHO = 180°$ (supplementary angles) and $\angle OBA + \angle AHO = 180°$ (opposite angles of concyclic quadrilateral), so that $\angle PHA = \angle OBA$. Next, in triangle AOB with $AO = OB$,

$$\angle OBA = \frac{1}{2}(180° - \angle AOB) = 90° - \frac{1}{2}\angle AOB = 90° - \angle C = \angle PAH \ .$$

So, $\angle PHA = \angle PAH$; thus, triangle APH is isosceles and $AP = PH$. Similarly, $QB = QH$. Therefore $PQ = PH + QH = AP + BQ$ as desired. ♠