Solutions to the May problems.

381. Determine all polynomials $f(x)$ such that, for some positive integer k,

$$f(x^k) - x^3 f(x) = 2(x^3 - 1)$$

for all values of x.

Solution. If $f(x)$ is constant, then $f(x) \equiv -2$. Suppose that $f(x)$ is a nonconstant polynomial of positive degree d. Then the degree of the terms of the left side must be greater than 3, so that $f(x^k)$ and $x^3 f(x)$ must have the same leading term. Therefore $\deg f(x^k) = \deg x^3 f(x)$, so that $kd = 3 + d$ or $3 = (k-1)d$. Therefore, $(k, d) = (4, 1), (2, 3)$.

Suppose that $(k, d) = (4, 1)$. Since $f(0) = -2$, we must have $f(x) = ax - 2$ for some constant a. It is readily checked that this solution is valid for all values of a.

Suppose that $(k, d) = (2, 3)$. Then $x^3[f(x) + 2] = f(x^2) + 2$. ¿From this equation, we see that its two sides must have terms of degree at least 3 and only terms of even degree; thus, its two sides involve terms in x^4 and x^6. It follows that $f(x)$ must have constant term -2, no term in x and x^2 and a term in x^3. Thus, $f(x) = bx^3 - 2$. Again, it can be checked that any function of this form is valid.

Comment. Many solvers forgot to deal with the possibility of a constant function. Also, having gotten the forms $ax - 2$ and $bx^3 - 2$, one should check that they actually work.

382. Given an odd number of intervals, each of unit length, on the real line, let S be the set of numbers that are in an odd number of these intervals. Show that S is a finite union of disjoint intervals of total length not less than 1.

Solution 1. The proof is by induction on the odd number of intervals. The result is obvious if the set contains only one interval. Suppose that it holds when there are $2k - 1 \geq 1$ intervals. Let a set of $2k + 1$ intervals satisfying the condition be given. Let I and J be the two intervals whose left endpoints are least. Suppose that $I \cap J = K$. Note that the lengths of $I \backslash J$ and $J \backslash I$ are the same.

Suppose that the intervals I and J are removed from the set. Then, by the induction hypothesis, there is a finite union T of disjoint intervals of total length consisting of all points that lie in oddly many intervals apart from I and J. Restore the intervals I and J to form the set S. Outside of the union of I and J, the sets S and T agree. If I is the leftmost interval, then S includes $I \backslash J$ along with $K \cap T$. The only part of T that might not belong to S must lie within the set $J \backslash I$; but this is compensated by the inclusion of $I \backslash J$. The result follows.

Solution 2. [Y. Zhao] That S is a union of disjoint intervals can be established. Let $2n + 1$ intervals $I_0, I_1, \cdots, I_{2n}$ of unit length be given in increasing order of left endpoint. Define

$$f_i(x) = \begin{cases} 1, & \text{if } x \in I_i \ ; \\ 0, & \text{if } x \notin I_i \ . \end{cases}$$

for $0 \leq i \leq 2n$. Let

$$F(x) = \sum_{i=0}^{2n} (-1)^i f_i(x) \ .$$

Suppose that x is a real number in $I_0 \cup I_1 \cup \cdots \cup I_n$. Let j be the minimum index and k the maximum index of the intervals that contain x. Then $x \in I_i$ if and only if $j \leq i \leq k$, and so $F(x) = \sum_{i=j}^{k} (-1)^i$. The value of $|F(x)|$ is 0 if and only if there are an even number of summands, *i.e.* $k - j + 1$ is even and 1 if and only if there are an odd number of summands. If x belongs to none of the intervals, then $F(x) = 0$. Hence the length of S is equal to

$$\int_{-\infty}^{\infty} |F(x)|dx \geq \int_{-\infty}^{\infty} F(x)dx = \sum_{i=0}^{2n} (-1)^i \int_{-\infty}^{\infty} f_i(x)dx$$
$$= \sum_{i=0}^{2n} (-1)^i = 1$$

as desired.

383. Place the numbers $1, 2, \cdots, 9$ in a 3×3 unit square so that
(a) the sums of numbers in each of the first two rows are equal;
(b) the sum of the numbers in the third row is as large as possible;
(c) the column sums are equal;
(d) the numbers in the last row are in descending order.

Prove that the solution is unique.

Comment. The problem is not quite correct. The solution is unique up to the order of the first two rows. Most students picked this up.

Solution. The first two rows should contain six numbers whose sum S is as small as possible and is even. This sum is at least $1 + 2 + 3 + 4 + 5 + 6 = 21$, so the sum is at least 22.

If the sum of the first two rows is 22, then the entries must be $1, 2, 3, 4, 5, 7$. The row that contains 1 must contain 3 and 7. The column sums are each 15, so the column that contains 7 cannot contain 8 or 9, so must contain in its third row the number 6. Hence one of the columns consists of $7, 2, 6$. The column that contains 5 cannot contain 8, as the 2 has already been used in another column.

Taking the last row as $(9, 8, 6)$, we obtain the top two rows $(5, 4, 2)$ and $(1, 3, 7)$. This satisfies the conditions. Thus, we have a solution that minimizes the sum of the first two rows and maximizes the sum of the last row.

Comment. The last row sum cannot be more that $9 + 8 + 7$, and must be odd (45 minus the sum of the first two rows). So we can start with the last row as $(9, 8, 6)$ and work from there.

384. Prove that, for each positive integer n,

$$(3 - 2\sqrt{2})(17 + 12\sqrt{2})^n + (3 + 2\sqrt{2})(17 - 12\sqrt{2})^n - 2$$

is the square of an integer.

Solution. Oberve that

$$(1 \pm \sqrt{2})^2 = 3 \pm 2\sqrt{2}$$
$$(1 \pm \sqrt{2})^4 = (3 \pm 2\sqrt{2})^2 = 17 \pm 12\sqrt{2}$$

and

$$-1 = (1 + \sqrt{2})(1 - \sqrt{2}) \ .$$

The given expression is equal to

$$(1 + \sqrt{2})^{4n-2} + (1 - \sqrt{2})^{4n-2} + 2[(1 + \sqrt{2})(1 - \sqrt{2})]^{2n-1} = [(1 + \sqrt{2})^{2n-1} + (1 - \sqrt{2})^{2n-1}]^2 \ .$$

Since

$$(1 \pm \sqrt{2})^{2n-1} = \sum_{k=0}^{n-1} \binom{2n-1}{2k} 2^k \pm \sqrt{2} \sum_{k=0}^{n-1} \binom{2n-1}{2k+1} 2^k \ ,$$

the quantity in square brackets is the integer

$$\sum_{k=0}^{n-1} \binom{2n-1}{2k} 2^{k+1} \ .$$

The result follows.

385. Determine the minimum value of the product $(a+1)(b+1)(c+1)(d+1)$, given that $a, b, c, d \geq 0$ and

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} + \frac{1}{d+1} = 1 \ .$$

Solution 1. By the inequality of the harmonic and geometric means of the four quantities, we have that

$$[(a+1)(b+1)(c+1)(d+1)]^{1/4} \geq \left[\frac{1}{4} \left(\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} + \frac{1}{d+1} \right) \right]^{-1} = 4 \ ,$$

whence the product must be at least $4^4 = 256$. This bound is achieved when $a = b = c = d = 3$.

Solution 2. Let $u^4 = (a+1)(b+1)$, $v^4 = (c+1)(d+1)$. Then, by the Arithmetic-Geometric Means Inequality,

$$\begin{aligned} 1 &= \frac{a+b+2}{u^4} + \frac{c+d+2}{v^4} \\ &\geq \frac{2\sqrt{(a+1)(b+1)}}{u^4} + \frac{2\sqrt{(c+1)(d+1)}}{v^4} \\ &= \frac{2}{u^2} + \frac{2}{v^2} = \frac{2(u^2+v^2)}{u^2 v^2} \\ &\geq \frac{4uv}{u^2v^2} = \frac{4}{uv} \ , \end{aligned}$$

so that $uv \geq 4$. The result follows with equality when $a = b = c = d = 3$.

386. In a round-robin tournament with at least three players, each player plays one game against each other player. The tournament is said to be *competitive* if it is impossible to partition the players into two sets, such that each player in one set beat each player in the second set. Prove that, if a tournament is not competitive, it can be made so by reversing the result of a single game.

Solution. In the following solutions, let $a > b$ denote that player a beats (wins over) player b. Note that this relation is *not* transitive, *i.e.* $a > b$ and $b > c$ does not necessarily imply that $a > c$.

Solution 1. [A. Wice] Suppose there are n players. We establish two preliminary results:

(1) The players can be labelled so that $a_1 > a_2 > \cdots > a_n$;

(2) If the players can be labelled to form a loop or circuit, thus $a_1 > a_2 > \cdots > a_n > a_1$, then the tournament is competitive.

Statement (1) can be established by induction. (If it holds for all tournaments with fewer than n players, then consider any player z in the tournament with n players. The players that beat z can be formed into a line as can those whom z beats. Now insert the player between the two lines to get a line of players each beating the next.) For statement (2), suppose if possible there are two nonvoid sets A and B partitioning all the players such that each player in A beats each player in B. Any player beaten by a player in B must also lie in B. Suppose we have a loop as in Statement (2). If, say, a_i belongs to B, then so does a_{i+1} and so on all around the loop to a_{i-1}, yielding a contradiction.

Suppose that we have a non-competitive tournament with sets A and B as above. Let $a_1 > a_2 > \cdots > a_k$ and $b_1 > b_2 > \cdots > b_m$ be labellings of the players in A and B as permitted in result (1). We also have

$a_k > b_1$ and $a_1 > b_m$. Reverse the game involving a_1 and b_m to make $b_m > a_1$. By result (2), we now have a tournament that is competitive.

Solution 2. Suppose that we are given a noncompetitive tournament T, and that the players are partitioned into two sets A and B for which each player in A beats each player in B. Suppose a is a player in A who loses to the smallest number of competitors in A; let A_1 be the subset of those in A who beat a. Suppose that b is a player in B who wins against the smallest number of players in B; let B_1 be the subset of B who loses to b.

In T, $a > b$. Form a new tournament T' from T by switching the result of the game between a and b, so that $b > a$ and otherwise the results in T and T' are the same. Suppose, if possible, that T' is noncompetitive. Then we can partition the set of players into two subsets U and V, for which each player in U beats each player in V.

Suppose that $a \in V$. Since a beat every player in B besides B, we must have that $U \cap B \subseteq \{b\}$, so that $U \subseteq A \cup \{b\}$. Indeed, $U \subseteq A_1 \cup \{b\}$, so that $A \backslash A_1 \subseteq V$. Consider a player x in U. This player lies in A_1 and must beat every player in $A \backslash A_1$ as well as a, and lose only to other players in A_1, *i.e.*, to fewer players in A than a loses to. But this contradicts the definition of a. Therefore, $a \in U$, so that $b \in U$ as well, since $b > a$ in T'.

Since b is beaten by every player in A in T, $V \cap A \subseteq \{a\}$, so that $V \subseteq B \cup \{a\}$. Indeed, $V \subseteq B_1 \cup \{a\}$, so that $B \backslash B_1 \subseteq U$. Any player in B_1 can win only against other competitors in B_1, ı.e.. to fewer players in B than b beats, giving a contradiction.

Hence $U \cup V = \{a, b\}$, contradicting the fact that the tournament has at least three players.

Comment. Several solvers were too loose in determining the pair that ought to be switched. Not just any pair of players from A and B will do; they have to be carefully delineated. A good thing to do in such a problem is to have an example that you can test your argument against. For example, consider the following tournament with four players a, b, c, d for which $a > c$, $a > d$, $b > a$, $b > c$, $b > d$, $c > d$. This is a noncompetitive tournament for which we can take

$$(A, B) = (\{a, b, c\}, \{d\}) \quad \text{or} \quad (\{a.b\}, \{c, d\}) \quad \text{or} \quad (\{b\}, \{a, c, d\}),$$

The only game whose results can be reversed to give a noncompetitive tournament is that between b and d, which will result in the cycle $a > c > d > b > a$. The other reversals result in competitive tournaments: (1) $c > a$, $(A, B) = (\{b, c\}, \{a, d\})$; (2) $a > b$, $(A, B) = (\{a\}, \{b, c, d\})$; (3) $d > a$; $(A, B) = (\{b\}, \{a, c, d\})$; (4) $c > b$; $(A, B) = (\{a, b, c\}, \{d\})$: (5) $d > c$; $(A, B) = (\{a, b, d\}, \{c\})$.

387. Suppose that a, b, u, v are real numbers for which $av - bu = 1$. Prove that

$$a^2 + u^2 + b^2 + v^2 + au + bv \geq \sqrt{3} \ .$$

Give an example to show that equality is possible. (Part marks will be awarded for a result that is proven with a smaller bound on the right side.)

Solution 1. [C. Sun] Let $x = a^2 + b^2$, $y = u^2 + v^2$, $z = au + bv$. Then $xy = z^2 + 1$.

Observe that

$$(t\sqrt{3} + 1)^2 \geq 0 \Longrightarrow 3t^2 + 1 \geq -2t\sqrt{3} \Longrightarrow 4t^2 + 4 \geq (\sqrt{3} - t)^2 \ .$$

¿From this, we find that

$$\begin{aligned}(x+y)^2 \geq 4xy = 4(z^2+1) = 4z^2 + 4 &\geq (\sqrt{3} - z)^2 \\ &\Longrightarrow x + y \geq \sqrt{3} - z \\ &\Longrightarrow x + y + z \geq \sqrt{3}\end{aligned}$$

as desired.

Solution 2. [Y. Zhao] Note that

$$a^2 + u^2 + b^2 + v^2 + au + bv = \left(u + \frac{a}{2}\right)^2 + \left(v + \frac{b}{2}\right)^2 + \frac{3}{4}\left(a^2 + b^2\right) .$$

For each fixed a and b, the function is minimized when (u, v) is closest to $(\frac{a}{2}, \frac{b}{2})$. But (u, v) lies on the line $bx - ay + 1 = 0$, so the distance between (u, v) and $(-\frac{a}{2}, -\frac{b}{2})$ is at least equal to the distance from $(-\frac{a}{2}, -\frac{b}{2})$ to the line of equation $bx - ay + 1$, namely $(a^2 + b^2)^{-1/2}$. Hence

$$\left(u + \frac{a}{2}\right)^2 + \left(v + \frac{b}{2}\right)^2 + \frac{3}{4}\left(a^2 + b^2\right) \geq \frac{1}{a^2 + b^2} + \frac{3}{4}(a^2 + b^2) \geq \sqrt{3}$$

by the Arithmetic-Geometric Means Inequality. Equality occurs, for example, when

$$(a, b, u, v) = \left(\frac{2^{1/2}}{3^{1/4}}, 0, \frac{-1}{2^{1/2}3^{1/4}}, \frac{3^{1/4}}{2^{1/2}}\right) .$$

Solution 3. [G. Ghosn] We use a vector argument, with boldface characters denoting vectors. Let $\mathbf{a} = (a, b)$, $\mathbf{u} = (u, v)$ and $\mathbf{v} = (v, -u)$. It is given that $\mathbf{a} \cdot \mathbf{v} = 1$. Since $\mathbf{u}$ and $\mathbf{v}$ form a basis for which $\mathbf{u} \cdot \mathbf{v} = 0$, $\{\mathbf{u}, \mathbf{v}\}$ is an orthogonal basis for two-dimensional Euclidean space. Hence there are scalars α and β for which $\mathbf{a} = \alpha\mathbf{u} + \beta\mathbf{v}$. Taking the inner (dot) product of this equation with $\mathbf{v}$ yields $\beta = |\mathbf{v}|^{-2} = u^2 + v^2$.

We have that $a^2 + b^2 = \mathbf{a} \cdot \mathbf{a} = (\alpha^2 + \beta^2)(u^2 + v^2)$ and $\alpha u + \beta v = \mathbf{a} \cdot \mathbf{u} = \alpha(u^2 + v^2)$. Hence

$$\begin{aligned} a^2 + u^2 + b^2 + v^2 + au + bv &= (\alpha^2 + \beta^2 + 1 + \alpha)(u^2 + v^2) \\ &= \frac{1}{4}[(2\alpha + 1)^2 + 3 + 4\beta^2](u^2 + v^2) \\ &\geq \frac{3}{4}(u^2 + v^2) + \frac{1}{u^2 + v^2} \geq 2\left(\frac{\sqrt{3}}{2}\right)(u^2 + v^2)\left(\frac{1}{u^2 + v^2}\right) = \sqrt{3} , \end{aligned}$$

by the Arithmetic-Geometric Means Inequality, with equality if and only if $\alpha = -1/2$ and $u^2 + v^2 = 2/\sqrt{3}$. We can achieve equality with

$$(a, b, u, v) = \left(\frac{3^{1/4}}{2^{1/2}}, \frac{-1}{2^{1/2}3^{1/4}}, 0, \frac{2^{1/2}}{3^{1/4}}\right) .$$

Solution 4. [A. Wice] Let $\mathbf{a} = (a, b)$ and $\mathbf{u} = (u, v)$, and let θ be the angle between the vectors $\mathbf{a}$ and $\mathbf{u}$. The area of the parallelogram with sides $\mathbf{a}$ and $\mathbf{u}$ is equal to

$$|\mathbf{a} \times \mathbf{u}| = |\mathbf{a}||\mathbf{u}| \sin\theta = |av - bu| = 1 .$$

Observe that $0 < \theta < 180°$. We have that

$$\begin{aligned} a^2 + u^2 + b^2 + v^2 + au + bv &= |\mathbf{a}|^2 + |\mathbf{u}|^2 + \mathbf{a} \cdot \mathbf{u} \\ &= |\mathbf{a}|^2 + |\mathbf{u}|^2 + |\mathbf{a}||\mathbf{u}| \cos\theta \\ &\geq |\mathbf{a}||\mathbf{u}|(2 + \cos\theta) = \frac{2 + \cos\theta}{\sin\theta} , \end{aligned}$$

by the Arithmetic-Geometric Means Inequality.

Now

$$\begin{aligned} 1 &\geq -\cos(\theta + 60°) = -\frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta \\ &\Longrightarrow 2 + \cos\theta \geq \sqrt{3}\sin\theta \\ &\Longrightarrow \frac{2 + \cos\theta}{\sin\theta} \geq \sqrt{3} , \end{aligned}$$

with equality if and only if $\theta = 120°$. Hence

$$a^2 + u^2 + b^2 + v^2 + au + bv \geq \sqrt{3}$$

with equality if and only if $|\mathbf{a}| = |\mathbf{u}| = 2^{1/2}/3^{1/4}$.

Hence we select

$$(a, b, u, v) = (k\cos\alpha, k\sin\alpha, k\cos(\alpha + 120°), k\sin(\alpha + 120°)$$

with $k = 2^{1/2}/3^{1/4}$ and some angle α. Taking $\alpha = 30°$ yields the example

$$(a, b, u, v) = \left(\frac{3^{1/4}}{2^{1/2}}, \frac{1}{3^{1/4} \cdot 2^{1/2}}, \frac{-3^{1/4}}{2^{1/2}}, \frac{1}{3^{1/4} \cdot 2^{1/2}}\right) .$$

Solution 5. Let $a = p\cos\phi$, $b = p\sin\phi$, $u = q\cos\theta$ and $v = q\sin\theta$, where p and q are positive reals. Then $1 = av - bu = pq\sin(\theta - \phi)$, from which we deduce that $pq \geq 1$ and that $\cos^2(\theta - \phi) = (p^2q^2 - 1)/(p^2q^2)$. Therefore

$$\begin{aligned} a^2 + u^2 + b^2 + v^2 + au + bv &= p^2 + q^2 + pq\cos(\theta - \phi) \\ &\geq p^2 + q^2 - pq\sqrt{(p^2q^2 - 1)/(p^2q^2)} = p^2 + q^2 - \sqrt{p^2q^2 - 1} \\ &\geq 2pq - \sqrt{p^2q^2 - 1} \ , \end{aligned}$$

by the Arithmetic-Geometric Means Inequality.

We need to show that $2t - (t^2 - 1)^{1/2} \geq 3^{1/2}$ for $t \geq 1$, or equivalently,

$$4t^2 + t^2 - 1 - 4t\sqrt{t^2 - 1} \geq 3 \Longleftrightarrow 5t^2 - 4 \geq 4t\sqrt{t^2 - 1} \ .$$

This in turn is equivalent to $25t^4 - 40t^2 + 16 \geq 16t^4 - 16t^2$ which reduces to the true inequality $(3t^2 - 4)^2 \geq 0$. The minimum of the left member of the inequality occurs when $t = 2/\sqrt{3}$ and $\cos^2(\theta - \phi) = (t^2 - 1)/t^2 = 1/4$.

Taking $p = q = 2^{1/2}3^{-1/4}$. $\phi = 150°$ and $\theta = 30°$ yields the example in Solution 4.

Solution 6. [C. Bao] This solution uses Lagrange Multipliers. Let

$$F(a, b, u, v, \lambda) = a^2 + u^2 + b^2 + v^2 + au + bv - \lambda(av - bu - 1) \ .$$

Then, the Lagrange conditions become

$$0 = \frac{\partial F}{\partial a} = 2a + u - \lambda v$$

$$0 = \frac{\partial F}{\partial b} = 2b + v + \lambda u$$

$$0 = \frac{\partial F}{\partial u} = 2u + a + \lambda b$$

$$0 = \frac{\partial F}{\partial v} = 2v + b - \lambda a$$

from which we obtain that

$$3(a + u) + \lambda(b - v) = 0 = (b - v) + \lambda(a + u) \ .$$

Therefore $3(a + u) = \lambda^2(a + u)$, so that, either $\lambda = \pm \pm \sqrt{3}$ or $a + u = b - v = 0$ at a critical point.

Suppose, first, that $\lambda^2 = 3$. Then

$$2a^2 + au + au + 2u^2 = \lambda(av - bu) \Longrightarrow \lambda = 2(a^2 + u^2 + au) \ ,$$

and

$$2v^2 + bv + vb + 2b^2 = \lambda(av - bu) \Longrightarrow \lambda = 2(b^2 + v^2 + bv) \ .$$

Therefore, at a critical point,

$$a^2 + u^2 + b^2 + v^2 + au + bv = \lambda \ .$$

Since, double the left side is equal to $(a+u)^2 + (b+v)^2 + a^2 + u^2 + b^2 + v^2$, we must have that $\lambda = \sqrt{3}$.

At this point in the argument, a technical difficulty arises, as it must be argued somehow that the critical point is a minimum, rather than a maximum or a saddle point. One way to do this is to establish that the objective function becomes infinite when we move towards infinity on the constraint surface, that it must attain a minimum value on the constraint surface (which requires a compactness argument) and use the fact that a single value of λ is turned up for a critical point.

The second possibility is that $a + u = b - v = 0$. Since $av - bu = 1$, this leads to $uv = -1/2$ and

$$a^2 + u^2 + b^2 + v^2 + au + bv = u^2 + 3v^2 \geq 2\sqrt{3}|uv| = \sqrt{3}$$

at the critical points.

Comment. This was not an easy problem and I garnered a larger collection of nice solutions than I expected. The lower bound of 2 is easily obtained by noting that

$$\begin{aligned}
&2[a^2 + u^2 + b^2 + v^2 + au + bv] \\
&\qquad = [a^2 + u^2 + b^2 + v^2 + 2au + 2bv] + [a^2 + u^2 + b^2 + v^2 + 2bu - 2av] + 2 \\
&\qquad = (a+u)^2 + (b+v)^2 + (b+u)^2 + (a-v)^2 + 2 \geq 2 \ .
\end{aligned}$$

Equality would require that $a = v = -u$ and $b = -u = -v$, which cannot be realized simultaneously.