Solutions to October problems

409. Find the number of ways of dealing n cards to two persons ($n \geq 2$), where the persons may receive unequal (positive) numbers of cards. Disregard the order in which the cards are received.

Solution. If we allow hands with no cards, there are 2^n ways in which they may be dealt (each card may go to one of two people). There are two cases in which a person gets no cards. Subtracting these gives the result: $2^n - 2$.

410. Prove that $\log n \geq k \log 2$, where n is a natural number and k the number of distinct primes that divide n.

Solution. Let n be a natural number greater than 1 and $p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ its prime factorization. Since $p_i \geq 2$ and $a_i \geq 1$ for all i,

$$n \geq 2^{a_1+a_2+\cdots+a_k} \geq 2^k \ .$$

This is also true for $n = 1$, for in this case, $k = 0$ and $n = 2^0$. Thus, for any base b exceeding 1,

$$\log_b n \geq \log_b 2^k = k \log_b 2 \ .$$

411. Let b be a positive integer. How many integers are there, each of which, when expressed to base b, is equal to the sum of the squares of its digits?

Solution. A simple calculation shows that 0 and 1 are the only single-digit solutions. We show that there are no solutions with three or more digits. Suppose that $n = a_0 + a_1 b + \cdots + a_m b^m$ where $m \geq 2$, $1 \leq a_m \leq b - 1$ and $0 \leq a_i \leq b - 1$ for $0 \leq i \leq m - 1$. Then

$$\begin{aligned}(a_0 + a_1 b + \cdots + a_m b^m) - (a_0^2 + a_1^2 + \cdots + a_m)^2 \\ &= a_1(b - a_1) + a_2(b^2 - a_2) + \cdots + a_m(b^k - a_m) - a_0(a_0 - 1) \\ &\geq a_m(b^m - a_m) - a_0(a_0 - 1) \geq 1 \cdot (b^2 - (b-1)) - (b-1)(b-2) \\ &= 2b - 1 \geq 0 \ .\end{aligned}$$

Thus, there are at most two digits for any example.

Let $N(b)$ denote the total number of solutions, and $N_2(b)$ the number of two digit solutions. Thus, $N(n) = N_2(b) + 2$.

Thus, $N_2(n)$ is the number of pairs (a_0, a_1) satisfying

$$a_0 + a_1 b = a_0^2 + a_1^2 \ , \qquad 0 \leq a_0 < b, 1 \leq a_1 < b \ . \tag{1}$$

The transformation given by $2a_0 = p + 1$, $2a_1 = b + q$ establishes a one-one correspondence between the pairs (a_0, a_1) satisfying (1) and the pairs (p, q) satisfying

$$p^2 + q^2 = 1 + b^2 \ , \qquad p \text{ odd } , 3 \leq p \leq b, 1 \leq q \leq b \ . \tag{3}$$

Now we can express the number of solutions of (2) in terms of the number $r(k)$ of solutions to

$$c^2 + d^2 = k \ . \tag{3}$$

Suppose that b is even. Then $1 + b^2$ is odd, so that exactly one of p or q is odd. Thus, given a solution (p, q) to (2) we can generate three others that solve (3) via $(c, d) = (-p, q), (q, p), (q - p)$. We also add the eight remaining solutions $(\pm 1, \pm b)$ and $(\pm b, \pm 1)$. This shows that $r(1 + b^2) = 4N_2(b) + 8 = 4N(b)$.

Suppose that b is odd. Then $1 + b^2 \equiv 2 \pmod 4$; hence, both p and q must be odd. Thus, from any solution (p, q) to (2) we can generate another solution to (3) via $(c, d) = (-p, q)$. We also add the remaining four uncounted solutions, $(\pm 1, \pm b)$. This shows that $r(1 + b^2) = 2N_2(b) + 4 = 2N(b)$.

The quantity $r(k)$ can be computed from a formula given, for example, in the book *Introduction to the Theory of Numbers* by Hardy and Wright. Using the fact that no prime of the form $4j + 3$ can divide $1 + b^2$, we find that

$$r(1+b^2) = \begin{cases} 4\tau(1+b^2)\ , & \text{if } b \text{ is even,} \\ 2\tau(1+b^2)\ , & \text{if } b \text{ is odd,} \end{cases}$$

where $\tau(n)$ is the number of positive integer divisors of n. Thus $N(b) = \tau(1 + b^2)$.

412. Let A and B be the midpoints of the sides, EF and ED, of an equilateral triangle DEF. Extend AB to meet the circumcircle of triangle DEF at C. Show that B divides AC according to the golden section. (That is, show that $BC : AB = AB : AC$.)

Solution. Consider the chords ED and CC'. The angles EBC' and CBD are equal, since they are vertically opposite, while angles $C'ED$ and DCC' are equal since they are subtended by the same chord $C'D$. Thus triangles $C'EB$ and DCB are similar. Therefore $EB : C'B = BC : BD$.

Since $EB = BD = AB$,

$$BC : AB = BC : BD = EC : C'B = AB : AC\ .$$

413. Let I be the incentre of triangle ABC. Let A', B' and C' denote the intersections of AI, BI and CI, respectively, with the incircle of triangle ABC. Continue the process by defining I' (the incentre of triangle $A'B'C'$), then $A''B''C''$, *etc.*. Prove that the angles of triangle $A^{(n)}B^{(n)}C^{(n)}$ get closer and closer to $\pi/3$ as n increases.

Solution. From triangle IAC we have that $\angle AIC = \pi - \frac{A}{2} - \frac{C}{2} = \frac{\pi+B}{2}$, so that $B' = \angle A'B'C' = \frac{1}{2}\angle A'IC' = \frac{1}{2}\angle AIC = \frac{\pi+B}{4}$. Similar relations hold for A' and C'. Assuming, wolog, $A \le B \le C$, then $A' = \frac{1}{4}(\pi + A) \le B' \le \frac{1}{4}(\pi + B) \le C' = \frac{1}{4}(\pi + C)$, and $C' - A' = \frac{1}{4}(C - A)$, so that triangle $A'B'C'$ is "four times closer" to equilateral than triangle ABC is. The result follows.

414. Let $f(n)$ be the greatest common divisor of the set of numbers of the form $k^n - k$, where $2 \le k$, for $n \ge 2$. Evaluate $f(n)$. In particular, show that $f(2n) = 2$ for each integer n.

Solution. For any prime p, $f(n)$ cannot contain a factor p^2 because $p^2 \not| k(k^{n-1} - 1)$ for $k = p$. For any n, $2|f(n)$.

If p is an odd prime and if a is a primitive root modulo p, then $p|a(a^{n-1} - 1)$ only if $(p-1)|(n-1)$. On the other hand, if $(p-1)|(n-1)$, then $p|(k^n - k)$ for every k. Thus, if P_n is the product of the distinct odd primes p for which $(p-1)|(n-1)$, then $f(n) = 2P_n$. (In particular, $6|f(n)$ for every odd n.)

As $p - 1$ is not a divisor of $2n - 1$ for any odd prime p, it follows that $f(n) - 2$.

Comments. The symbol $|$ means "divides" or "is a divisor of". For every prime p, there is a number a (called the *primitive root* modulo p such that $p - 1$ is the smallest values of k for which $a^k \equiv 1$ modulo p.

415. Prove that

$$\cos\frac{\pi}{7} = \frac{1}{6} + \frac{\sqrt{7}}{6}\left(\cos\left(\frac{1}{3}\arccos\frac{1}{2\sqrt{7}}\right) + \sqrt{3}\sin\left(\frac{1}{3}\arccos\frac{1}{2\sqrt{7}}\right)\right)\ .$$

Solution. The identity

$$\cos 7\theta = (\cos\theta + 1)(8\cos^3\theta - 4\cos^2\theta - 4\cos\theta + 1)^2 - 1$$

(derive this using de Moivre's theorem, or otherwise) implies that the three roots of $f(x) = 8x^3 - 4x^2 - 4x + 1$ are $\cos\frac{\pi}{7}$, $\cos\frac{3\pi}{7}$ and $\cos\frac{5\pi}{7}$. Observe that $\cos\frac{\pi}{7} > \cos\frac{3\pi}{7} > 0 > \cos\frac{5\pi}{7}$. Thus, $\cos\frac{\pi}{7}$ is the only root of the cubic polynomial $f(x)$ greater than $\cos\frac{3\pi}{7}$.

Let

$$a = \cos\left(\frac{1}{3}\arccos\frac{-1}{2\sqrt{7}}\right) ,$$

and let

$$\begin{aligned} c &= \frac{1}{6} + \frac{\sqrt{7}}{6}\left(\cos\left(\frac{1}{3}\arccos\frac{1}{2\sqrt{7}}\right) + \sqrt{3}\cos\left(\frac{1}{3}\arccos\frac{1}{2\sqrt{7}}\right)\right) \\ &= \frac{1}{6} + \frac{\sqrt{7}}{3}\cos\left(\frac{1}{3}\left(\pi - \arccos\frac{1}{2\sqrt{7}}\right)\right) \\ &= \frac{1}{6} + \frac{\sqrt{7}}{3}a \ . \end{aligned}$$

The function $g(x) = \cos(\frac{1}{3}\arccos x)$ is increasing for $-1 \le x \le 1$, so that $a > \cos(\frac{1}{3}\arccos(-1)) = \frac{1}{2}$. Therefore

$$x > \frac{1+\sqrt{7}}{6} > \frac{1}{2} > \cos\frac{3\pi}{7} \ .$$

Since $6c - 1 = 2\sqrt{7}a$, the identity $4\cos^3\theta - 3\cos\theta = \cos 3\theta$ gives

$$\frac{1}{14\sqrt{7}}(6c-1)^3 - \frac{3}{2\sqrt{7}}(6c-1) = \frac{-1}{2\sqrt{7}} \ .$$

Hence

$$f(c) = \frac{14\sqrt{7}}{27}\left(\frac{1}{14\sqrt{7}}(6c-1)^3 - \frac{3}{2\sqrt{7}}(6c-1) + \frac{1}{2\sqrt{7}}\right) = 0 \ ,$$

and so $c = \cos\frac{\pi}{7}$.