Problems for January, 2006

Please send your solution to

Prof. Edward J. Barbeau
Department of Mathematics
University of Toronto
40 St. George Street, Room 6290
Toronto, ON M5S 2E4

no later than February 28, 2006 It is important that your complete mailing address and your email address appear on the front page. If you do not write your family name last, please underline it.

423. Prove or disprove: if x and y are real numbers with $y \geq 0$ and $y(y+1) \leq (x+1)^2$, then $y(y-1) \leq x^2$.

424. Simplify

$$\frac{x^3 - 3x + (x^2-1)\sqrt{x^2-4} - 2}{x^3 - 3x + (x^2-1)\sqrt{x^2-4} + 2}$$

to a fraction whose numerator and denominator are of the form $u\sqrt{v}$ with u and v each linear polynomials. For which values of x is the equation valid?

425. Let $\{x_1, x_2, \cdots, x_n, \cdots\}$ be a sequence of nonzero real numbers. Show that the sequence is an arithmetic progression if and only if, for each integer $n \geq 2$,

$$\frac{1}{x_1 x_2} + \frac{1}{x_2 x_3} + \cdots + \frac{1}{x_{n-1} x_n} = \frac{n-1}{x_1 x_n} .$$

426. (a) The following paper-folding method is proposed for trisecting an acute angle.

(1) transfer the angle to a rectangular sheet so that its vertex is at one corner P of the sheet with one ray along the edge PY; let the angle be XPY;

(2) fold up PY over QZ to fall on RW, so that $PQ = QR$ and $PY \| QZ \| RW$, with QZ between PY and RW;

(3) fold across a line AC with A on the sheet and C on the edge PY so that P falls on a point P' on QZ and R on a point R' on PX;

(4) suppose that the fold AC intersects the fold QZ at B and carries Q to Q'; make a fold along BQ'.

It is claimed that the fold BQ' passes through P and trisects angle XPY.

Explain why the fold described in (3) is possible. Does the method work? Why?

(b) What happens with a right angle?

(c) Can the method be adapted for an obtuse angle?

427. The radius of the inscribed circle and the radii of the three escribed circles of a triangle are consecutive terms of a geometric progression. Determine the largest angle of the triangle.

428. **a**, **b** and **c** are three lines in space. Neither **a** nor **b** is perpendicular to **c**. Points P and Q vary on **a** and **b**, respectively, so that PQ is perpendicular to **c**. The plane through P perpendicular to **b** meets **c** at R, and the plane through Q perpendicular to **a** meets **c** at S. Prove that RS is of constant length.

429. Prove that

$$\sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \binom{kn}{n} = (-1)^{n+1} n^n .$$