Solutions for April

437. Let a, b, c be the side lengths and m_a, m_b, m_c the lengths of their respective medians, of an arbitrary triangle ABC. Show that

$$\frac{3}{4} < \frac{m_a + m_b + m_c}{a + b + c} < 1 \ .$$

Furthermore, show that one cannot find a smaller interval to bound the ratio.

Solution. We use the property that the intersection of the medians trisects the medians. From the triangle inequality, we obtain

$$a < \frac{2}{3}m_b + \frac{2}{3}m_c \qquad b < \frac{2}{3}m_c + \frac{2}{3}m_a \qquad c < \frac{2}{3}m_a + \frac{2}{3}m_b \ .$$

Summing these inequalities and manipulating gives the left inequality.

For the right inequality, extend a median, say m_b to the same length on the other side of AC. Joining A and C to the endpoint D of the extended median gives a parallellogram $ABCD$ with diagonal AD of length $2m_b$ and sides a and c. Do the same with the other two medians. ¿From the triangle inequality, we have that

$$2m_b < a + c \qquad 2m_c < a + b \qquad 2m_a < b + c \ .$$

Summing the inequalities will lead to the right inequality.

To see that the inequality cannot be improved, consider the isosceles triangle ABC with sides AB and AC of length 1 and angle A equal to 2θ, where $0 < \theta < 90°$. Then

$$(a, b, c) = (2\sin\theta, 1, 1)$$

and

$$(m_a, m_b, m_c) = (\cos\theta, f(\theta), f(\theta))$$

where $f(\theta) = \frac{1}{2}\sqrt{5 - 4\cos 2\theta}$. Observe that

$$\lim_{\theta\to 0} f(\theta) = \frac{1}{2} \qquad \text{and} \qquad \lim_{\theta\to 90°} f(\theta) = \frac{3}{2} \ .$$

We have that

$$\frac{m_a + m_b + m_c}{a + b + c} = \frac{\cos\theta + 2f(\theta)}{2(1 + \sin\theta)} \ .$$

When θ is close to $0°$, this ratio is close to 1, and when θ is close to $90°$, the ratio is close to $3/4$.

438. Determine all sets (x, y, z) of real numbers for which

$$x + y = 2 \qquad \text{and} \qquad xy - z^2 = 1 \ .$$

Solution. From the second equation, $z^2 = xy - 1 = x(2 - x) - 1 = -(x - 1)^2$. Since squares are nonnegative, we must have $z = 0 = (x - 1)$, so that $(x, y, z) = (1, 1, 0)$.

439. A natural number n, less than or equal to 500, has the property that when one chooses a number m randomly among $\{1, 2, 3, \cdots, 500\}$, the probability that m divides n (*i.e.*, n/m is an integer) is $1/100$. Find the largest such n.

Solution. The number n must have 5 divisors. If the prime factorization of n is $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$, then n has $(\alpha_1 + 1)(\alpha_2 + 1)\cdots(\alpha_k + 1)$ divisors. To obtain five divisors, n must have the form p^4 for some prime p. The largest such n is therefore $n = 3^4 = 81$, as $5^4 > 500$.

440. You are to choose 10 distinct numbers from $\{1, 2, 3, \cdots, 2006\}$. Show that you can choose such numbers with a sum greater than 10039 in more ways than you can choose such numbers with a sum less than 10030.

Solution. Let M be the set $\{1, 2, 3, \cdots, 2006\}$, let S be the set of 10−tuples of distinct elements of M with sum less than 10030, and L be the set of 10−tuples of distinct elements of M with sum greater than 10039. Define a function f on S by

$$f(a_1, a_2, \cdots, a_{10}) = (2007 - a_1, 2007 - a_2, \cdots, 2007 - a_{10}) \ .$$

Each $f(a_1, a_2, \cdots, a_{10})$ consists of ten distinct numbers in M and the sum of the numbers is

$$10 \cdot 2007 - \sum_{i=1}^{10} a_i > 27000 - 10030 = 10040 \ .$$

Hence the range of f is a subset of L. Since $(999, 1000, 1001, 1002, 1003, 1005, 1006, 1007, 1008, 1009)$ is in L, but not in the range of f, the range of f is a proper subset of L. As f is injective and the sets are finite, it follows that S has fewer elements than L.

441. Prove that, no matter how 15 points are placed inside a circle of radius 2 (including the boundary), there exists a circle of radius 1 (including the boundary) containing at least 3 of the 15 points.

Solution. We will cover the circle of radius 2 entirely with seven circles of radius 1. By the Pigeonhole Principle, at least one of these circles will contain at least three of the fifteen points.

Construct the circle of radius 1 concentric with the circle of radius 2; denote the centre of these circles by O. For each of the six 60°−sectors, construct a circle as follows. Let A and B be on the inner circle with $\angle AOB = 60^\circ$, and let OA and OB produced meet the outer circle at C and D respectively; let E be the midpoint of CD. The following triangles are equilaterial with side length 1: BOA, EAC, EBA, EDB. The inner circle and the circle with centre E and radius 1 together cover the sector COD.

442. Prove that the regular tetrahedron has minimum diameter among all tetrahedra that circumscribe a given sphere. (The diameter of a tetrahedron is the length of its longest edge.)

Solution. Let T be the tetrahedron with volume V and surface area S; suppose that r is the radius of the sphere inscribed within T. Let one vertex of T be at the origin and let the other three vertices and the centre of the sphere be given by the position vectors A, B, C and P, respectively. Then $P = \alpha A + \beta B + \gamma C$, with $\alpha, \beta, \gamma > 0$ and $\alpha + \beta + \gamma < 1$.

Suppose that $A \times B$ points into T. Then, since P can be written as the orthogonal sum of a vector in the plane of A and B and a vector of length r perpendicular to this plane from the point of tangency of the insphere,

$$r|A \times B| = P \cdot (A \times B) = \gamma C \cdot (A \times B) = 6\gamma V \ ,$$

and likewise

$$r|B \times C| = P \cdot (B \times C) = 6\alpha V \ ,$$

$$r|C \times A| = P \cdot (C \times A) = 6\beta V \ ,$$

$$r|(C - A) \times (B - A)| = (P - A) \cdot ((C - A) \times (B - A)) \ .$$

Since $P - A = (\alpha + \beta + \gamma - 1)A + \beta(B - A) + \gamma(C - A)$,

$$\begin{aligned}(P - A) \cdot ((C - A) \times (B - A)) &= (\alpha + \beta + \gamma - 1)A \cdot ((C - A) \times (B - A)) \\ &= (1 - \alpha - \beta - \gamma)(O - A) \cdot ((C - A) \times (B - A) \\ &= 6(1 - \alpha - \beta - \gamma)V \ .\end{aligned}$$

The total surface area S is equal to

$$\frac{1}{2}[|A \times B| + |B \times C| + |C \times A| + |(C-A) \times (B-A)|] = 3V/r$$

so that $r = 3V/A$. The triangle of given perimeter with maximum area is equilateral and (it is possible to show that) the tetrahedron of given surface area with maximum volume is regular. The desired result follows immediately from the last formula.

443. For $n \geq 3$, show that $n-1$ straight lines are sufficient to go through the interior of every square of an $n \times n$ chessboard. Are $n-1$ lines necessary?

Solution. Let the corners of the board be $(-n,n)$, $(0,n)$, $(-n,0)$, $(0,0)$. Draw $n-2$ lines L_i with slope $1/2$, the first intersecting the base line at $(-9/2,0)$, and the rest spaced so that the vertical distance between the lines is $3/2$ units. L_i has equation $y = \frac{1}{2}x + \frac{9}{4} + \frac{3i}{2}$, for $i = 0, 1, 2, \cdots, n-3$. (L_{i+1} is obtained by shifting L_i one unit to the left and then one unit up.) L_1 goes through the points $(-9/2,0)$, $(-4,1/4)$, $(-3,3/4)$, $(-5/2,1)$, $(-2,5/4)$, $(-1,7/4)$, $(-1/2,2)$, $(0,9/4)$. L_{n-3} goes through the point $(-n+9/2,n)$. (Thus, the arrangement is symmetric under 180 degree rotation.) Leaving out the two rightmost squares in the lowest row an the two leftmost squares in the top row, every unit square has an interior point on one or other of the lines. The $(n-1)$th line can be drawn to pass through the interiors of the four squares.

To prove necessity, we need an inductive argument. Let there be m rows and n columns with $m \leq n$, and let $f(m,n)$ be the minimum number of lines needed to cover an $m \times n$ board, in the sense that some interior point of every unit square is on one or another of the lines. Since any line going from one side to the other with ends in adjacent rows crosses both squares in a column exactly once, $f(1,n) = 1$, $f(2,n) = 2$, $f(n-1,n) \leq n-1$. It can be shown that $f(3,4) = 3$, $f(4,5) = 4$, $f(5,6) = 5$. Thus, $f(3,n) = 3$, $f(4,n) = 4$, $f(5,n) = 5$. If $f(n-1,n) = n-1$, then $f(m,n)$ is determined for all m,n: $f(n-k,n) = n-k$, if $0 < k < n$, $n \geq 3$; $f(n,n) = n-1$.