Solutions for January 2006

423. Prove or disprove: if x and y are real numbers with $y \geq 0$ and $y(y+1) \leq (x+1)^2$, then $y(y-1) \leq x^2$.

Solution 1. The statement is true. The result holds when $y \leq 1$ since $y(y-1) \leq 0 \leq x^2$. Assume $y \geq 1$, so that $\sqrt{y(y-1)} - 1 > 0$. We have that $4y(y+1) < (2y+1)^2$, so that $2\sqrt{y(y+1)} < 2y+1$. Hence

$$y(y-1) < y^2 + y - 2\sqrt{y(y+1)} + 1 = (\sqrt{y(y+1)} - 1)^2 \ .$$

Since $y(y+1) \leq (x+1)^2$ and $|x+1| \leq |x| + 1$, it follows that

$$y(y+1) \leq (|x|+1)^2 \Longrightarrow \sqrt{y(y+1)} - 1 \leq |x|$$

whence

$$y(y-1) < |x|^2 = x^2 \ .$$

Thus the assertion holds.

Solution 2. [D. Dziabenko] The statement holds. If $0 \leq y \leq 1$, then $y(y-1) \leq 0 \leq x^2$. Assume henceforth that $y > 1$. If $x + \frac{1}{2} \leq y$, then

$$y(y-1) = y(y+1) - 2y \leq (x+1)^2 - 2\left(x + \frac{1}{2}\right) = x^2 \ .$$

If $x + \frac{1}{2} > y$, then $x > y - \frac{1}{2} > 0$, whence

$$x^2 > \left(y - \frac{1}{2}\right)^2 = y(y-1) + \frac{1}{4} > y(y-1) \ .$$

Solution 3. [G. Ghosn] The result holds. Let $y > 0$. The region in the cartesian plane defined by $y(y+1) \leq (x+1)^2$ lies between the $x-$axis and the upper branch of the hyperbola with equation

$$\left(y + \frac{1}{2}\right)^2 - (x+1)^2 = \frac{1}{4} \ .$$

The region in the cartesian plane defined by $y(y-1) \leq x^2$ lies between the $x-$axis and the upper branch of the hyperbola with equation

$$\left(y - \frac{1}{2}\right)^2 - x^2 = \frac{1}{4}.$$

The second hyperbola lies above the first when $x = -1$. Thus, if it can be shown that the two hyperbolas do not intersect, then the top branch of the second hyperbola lies above top branch of the first, and the result will follow.

But solving $y(y+1) = (x+1)^2$ and $y(y-1) = x^2$ leads to $y = x + \frac{1}{2}$ and ultimately to $(x+\frac{1}{2})^2 - (x+\frac{1}{2}) = x^2$ which has no solution (being equivalent to $-1/4 = 0$).

424. Simplify

$$\frac{x^3 - 3x + (x^2-1)\sqrt{x^2-4} - 2}{x^3 - 3x + (x^2-1)\sqrt{x^2-4} + 2}$$

to a fraction whose numerator and denominator are of the form $u\sqrt{v}$ with u and v each linear polynomials. For which values of x is the equation valid?

Solution. For a real expression, we require that $x^2 \geq 2$. Observe that $x^3 - 3x + 2 = (x-1)^2(x+2)$ and that $x^3 - 3x - 2 = (x+1)^2(x-2)$. Thus, the denominator vanishes when $x = -2$, and we must exclude this value. Suppose, first, that $x \geq 2$. Then

$$\begin{aligned}\frac{x^3 - 3x + (x^2-1)\sqrt{x^2-4} - 2}{x^3 - 3x + (x^2-1)\sqrt{x^2-4} + 2} &= \frac{(x+1)^2(x-2) + (x^2-1)\sqrt{x^2-4}}{(x-1)^2(x+2) + (x^2-1)\sqrt{x^2-4}} \\ &= \frac{[(x+1)\sqrt{x-2}][(x+1)\sqrt{x-2} + (x-1)\sqrt{x+2}]}{[(x-1)\sqrt{x+2}][(x-1)\sqrt{x+2} + (x+1)\sqrt{x-2}]} \\ &= \frac{(x+1)\sqrt{x-2}}{(x-1)\sqrt{x+2}} \ .\end{aligned}$$

Now suppose that $x \leq -2$. Then

$$\begin{aligned}\frac{x^3 - 3x + (x^2-1)\sqrt{x^2-4} - 2}{x^3 - 3x + (x^2-1)\sqrt{x^2-4} + 2} &== \frac{-(x+1)^2(2-x) + (x^2-1)\sqrt{(2-x)(-2-x)}}{-(x-1)^2(-2-x) + (x^2-1)\sqrt{(2-x)(-2-x)}} \\ &= \frac{[(x+1)\sqrt{2-x}][-(x+1)\sqrt{2-x} + (x-1)\sqrt{-2-x}]}{[(1-x)\sqrt{-2-x}][(x-1)\sqrt{-2-x} - (x+1)\sqrt{2-x}]} \\ &= \frac{(x+1)\sqrt{2-x}}{(1-x)\sqrt{-2-x}} \ .\end{aligned}$$

Comment. Most solvers neglected to ensure that the quantities under the radical were nonnegative. This is the sort of "easy" question where many marks can be lost because of inattention to detail.

425. Let $\{x_1, x_2, \cdots, x_n, \cdots\}$ be a sequence of nonzero real numbers. Show that the sequence is an arithmetic progression if and only if, for each integer $n \geq 2$,

$$\frac{1}{x_1x_2} + \frac{1}{x_2x_3} + \cdots + \frac{1}{x_{n-1}x_n} = \frac{n-1}{x_1x_n} \ .$$

Solution. A constant sequence is an arithmetic progression and clearly satisfies the condition. Suppose that the sequence is a nonconstant arithmetic progression with common difference d. Then, for each positive index i, we have that

$$\frac{d}{x_ix_{i+1}} = \frac{x_{i+1} - x_i}{x_ix_{i+1}} = \frac{1}{x_i} - \frac{1}{x_{i+1}} \ .$$

Hence, for each $n \geq 2$,

$$\begin{aligned}d\sum_{i=1}^{n-1} \frac{1}{x_ix_{i+1}} &= \sum_{i=1}^{n-1} \left(\frac{1}{x_i} - \frac{1}{x_{i+1}}\right) \\ &= \left(\frac{1}{x_1} - \frac{1}{x_n}\right) = \frac{x_n - x_1}{x_1x_n} = \frac{(n-1)d}{x_1x_n} \ ,\end{aligned}$$

from which the condition follows.

On the other hand, suppose that the condition holds. Let $d = x_2 - x_1$, and suppose that we have established that $x_n - x_1 = (n-1)d$ (this is true for $n = 2$). Then, we have that

$$\frac{n}{x_1x_{n+1}} - \frac{n-1}{x_1x_n} = \frac{1}{x_nx_{n+1}}$$

so that

$$nx_n - (n-1)x_{n+1} = x_1 \Longrightarrow (n-1)(x_{n+1} - x_1) = n(x_n - x_1) = n(n-1)d$$

from which $x_{n+1} - x_1 = nd$. It follows, by induction, that the $\{x_n\}$ is an arithmetic progression.

426. (a) The following paper-folding method is proposed for trisecting an acute angle.

(1) transfer the angle to a rectangular sheet so that its vertex is at one corner P of the sheet with one ray along the edge PY; let the angle be XPY;

(2) fold up PY over QZ to fall on RW, so that $PQ = QR$ and $PY \| QZ \| RW$, with QZ between PY and RW;

(3) fold across a line AC with A on the sheet and C on the edge PY so that P falls on a point P' on QZ and R on a point R' on PX;

(4) suppose that the fold AC intersects the fold QZ at B and carries Q to Q'; make a fold along BQ'.

It is claimed that the fold BQ' passes through P and trisects angle XPY.

Explain why the fold described in (3) is possible. Does the method work? Why?

(b) What happens with a right angle?

(c) Can the method be adapted for an obtuse angle?

Solution. [F. Barekat] Let $\angle XPY = 3\theta$. Select B on QZ so that $\angle PBR = 4\theta$ and draw the circle with centre B that passes through P and R. Suppose that this circle intersects QZ at P' and XP at R'. Since $\angle PBR = 4\theta$, $\angle P'PY = \angle PP'Q = \frac{1}{2}\angle PP'R = \theta$. Also, $\angle P'RR' = \angle XPP' = \angle XPY - \angle P'PY = 2\theta$, so that $\angle R'RW = \angle P'RR' - \angle WRP' = \theta$.

Hence PP' and RR' are parallel chords in the circle, and their right bisectors pass through B and defines the required fold to interchange P and P', and R and R'. Since $BP = BP'$, $\angle BPY = \angle QBP = \angle BPP' + \angle BP'P = 2\angle BP'P = 2\theta$, so that $\angle BPR' = \theta$ and BP trisects the angle.

The fold (reflection) fixes B and interchanges P and P', and Q and Q'. Since P', B, Q are collinear, so are P, B, Q'. Hence the line through B and Q' also passes through P and so trisects the angle.

(b) When $\angle XPY = 90°$, then X lies on PR and R and R' coincide. We have that $PR = P'R = P'P$, so that triangle $PP'R$ is equilateral and RC is an altitude. Hence Q' is the midpoint of $P'R$ and $\angle Q'PP' = 30° = \angle P'PX$.

(c) One way to trisect an obtuse angle is to trisect its supplement, and subtract the result from $60°$..

427. The radius of the inscribed circle and the radii of the three escribed circles of a triangle are consecutive terms of a geometric progression. Determine the largest angle of the triangle.

Solution 1. [F. Barekat] Let r be the inradius and r_a, r_b, r_c the respective exradii of the circles touching the sides a, b, c. Let s be the semiperimeter and Δ the area of the triangle. Recall that

$$\Delta = rs = r_a(s-a) = r_b(s-b) = r_c(s-c)$$

[derive this], from which

$$\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{s-a}{\Delta} + \frac{s-b}{\Delta} + \frac{s-c}{\Delta} = \frac{s}{\Delta} = \frac{1}{r}$$

so that r is the smallest term in the geometric progression. Suppose that $A \leq B \leq C$. Then

$$r_a = s\tan(A/2) \leq r_b = s\tan(B/2) \leq r_c = s\tan(C/2) \ .$$

Hence there is a number $t > 1$ for which $r_a = tr$, $r_b = t^2 r$, $r_c = t^3 r$ and $s - b = t(s-c)$, $s - a = t^2(s-c)$, $s = t^3 s$. By Heron's formula,

$$\begin{aligned} t^3 r(s-c) &= r_c(s-c) = rs = \Delta \\ &= \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{t^6(s-c)^4} = t^3(s-c)^2 \ . \end{aligned}$$

Hence $r = s - c$, and so $\tan(C/2) = r/(s-c) = 1$. Thus $C = 90°$.

Solution 2. [D. Dziabenko] Define the symbols as in the first solution. The equation $r_a^{-1}+r_b^{-1}+r_c^{-1} = r^{-1}$ leads to $t^3 = t^2 + t + 1$. Now

$$\begin{aligned}\Delta &= \frac{1}{2}ab\sin C = \frac{1}{2}[(s-b+s-c)(s-a+s-c)\sin C \\ &= \frac{1}{2}[(t+1)(t^2+1)(s-c)^2\sin C] = t^3(s-c)^2\sin C\ .\end{aligned}$$

On the other hand, by Heron's formula, we find that

$$\Delta = \sqrt{t^6(s-c)^4} = t^3(s-c)^2\ .$$

Comparing the two expressions leads to $\sin C = 1$, so that $C = 90°$.

Solution 3. [G. Ghosn; A. Remorov] With $r \le r_a \le r_b \le r_c$ as in the previous solutions and from the inverse proportionality of $r : r_a : r_b : r_c$ and $s : (s-a) : (s-b) : (s-c)$, we have that $(s-a)(s-b) = s(s-c)$, whence

$$ab = (a+b-c)s \Longrightarrow 2ab = (a+b)^2 - c^2 \Longrightarrow c^2 = a^2 + b^2\ .$$

Hence the triangle is right, and its largest angle, C, is $90°$.

428. **a**, **b** and **c** are three lines in space. Neither **a** nor **b** is perpendicular to **c**. Points P and Q vary on **a** and **b**, respectively, so that PQ is perpendicular to **c**. The plane through P perpendicular to **b** meets **c** at R, and the plane through Q perpendicular to **a** meets **c** at S. Prove that RS is of constant length.

Solution. Let the point P on line **a** be given by $\mathbf{p} + s\mathbf{u}$ and the point Q on line **b** be given by $\mathbf{q} + t\mathbf{v}$, where $\mathbf{p}$, $\mathbf{q}$, $\mathbf{u}$ and $\mathbf{v}$ are fixed vectors and s, t parameters determining the points. Let $\mathbf{r}$ be a point on **c** and $\mathbf{w}$ be the direction vector for **c**. Wolog, we can normalize $\mathbf{u}$ and $\mathbf{v}$ so that $\mathbf{u}\cdot\mathbf{w} = \mathbf{v}\cdot\mathbf{w} = 1$.

By hypothesis, $[(\mathbf{p}+s\mathbf{u}) - (\mathbf{q}+t\mathbf{v}]\cdot\mathbf{w} = 0$, whence $s - t = (\mathbf{q}-\mathbf{p})\cdot\mathbf{w}$. Let R be given by $\mathbf{r}+m\mathbf{w}$ and S by $\mathbf{r}+n\mathbf{w}$. Then

$$[(\mathbf{p}+s\mathbf{u}) - (\mathbf{r}+m\mathbf{w})]\cdot\mathbf{v} = 0$$

so that $m = (\mathbf{p}-\mathbf{r})\cdot\mathbf{v} + s(\mathbf{u}\cdot\mathbf{v})$. Also

$$[(\mathbf{q}+t\mathbf{v}) - (\mathbf{r}+m\mathbf{w})]\cdot\mathbf{u} = 0$$

so that $n = (\mathbf{q}-\mathbf{r})\cdot\mathbf{u} + t(\mathbf{v}\cdot\mathbf{u})$. Hence

$$m - n = (\mathbf{p}-\mathbf{r}).\mathbf{v} - (\mathbf{q}-\mathbf{r}).\mathbf{u} + (\mathbf{u}\cdot\mathbf{v})[(\mathbf{q}-\mathbf{p}).\mathbf{w}$$

is independent of s and t and the result follows.

429. Prove that

$$\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}\binom{kn}{n} = (-1)^{n+1}n^n\ .$$

Solution 1. Let $f(x) = (1-(1+x)^n)^n$. Since $(1-(1+x)^n) = -nx + x^2g(x)$, for some polynomial $g(x)$, we have that

$$f(x) = (-1)^n n^n x^n + x^{n+1}h(x)$$

for some polymonial $h(x)$. On the other hand,

$$\begin{aligned} f(x) &= \sum_{k=0}^{n}(-1)^k\binom{n}{k}(1+x)^{nk} \\ &= \sum_{k=0}^{n}(-1)^k\binom{n}{k}\sum_{j=0}^{nk}\binom{nk}{j}x^j \\ &= \sum_{j=0}^{n^2}\left[\sum_{k=\lceil j/n\rceil}^{n}(-1)^k\binom{n}{k}\binom{nk}{j}\right]x^j \ . \end{aligned}$$

Comparing the coefficients of x^n in the two expressions for $f(x)$ yields that $(-1)^n n^n = \sum_{k=1}^{n}(-1)^k\binom{n}{k}\binom{nk}{n}$, from which the desired result follows.

Solution 2. Consider a set of n^2 distinct objects arranged in a $n \times n$ square array. There are n^n ways of choosing n of them so that one is chosen from each row. We count this in a different way, using the Principle of Inclusion-Exclusion. Let $f(r)$ be the number of ways of selecting the n objects so that they come from at most r distinct rows. There are $\binom{n}{r}$ ways of selecting the r rows containing the objects, and rn objects to choose from. Hence $f(r) = \binom{n}{r}\binom{rn}{n}$. [Note that this doublecounts choices involving fewer than r rows.]

There are $f(n) = \binom{n^2}{n}$ ways of choosing n objects from the array without restriction. But this includes the $f(n-1)$ selections where they are drawn from at most $n-1$ rows. But then $f(n) - f(n-1)$ subtracts off those from $n-2$ rows twice, so we need to add $f(n-2)$ back. But then, in $f(n)-f(n-1)+f(n-2)$, we have added in each selection from $n-3$ rows $\binom{3}{3}$ times in $f(n)$, subtracted in $\binom{3}{2}$ times in $f(n-1)$, added it back $\binom{3}{1}$ times. So we need to add it back. Continuing in this way, we find that

$$n^n = \sum_{k=0}^{n}(-1)^{n-k}f(k) = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{kn}{k}$$

from which the desired result follows.

Solution 4. [G. Ghosn] Let $P(x) = (x^n-1)^n = \sum_{k=0}^{n}\binom{n}{k}x^{kn}(-1)^{n-k}$. We calculate the nth derivative of the two expressions.

Recall Leibniz' Rule that $D^n(QR) = \sum_{k=0}^{n}\binom{n}{k}D^{n-k}(Q)D^k(R)$. Taking $Q(x) = (x-1)^n$ and $R(x) = (x^{n-1}+x^{n-2}+\cdots+1)^n$, we have that

$$\begin{aligned} D^n[(x^n-1)^n] &= D^n[(x-1)^n(x^{n-1}+\cdots+x+1)^n] \\ &= n!(x^{n-1}+\cdots+x+1)^n + (x-1)S(x) \ , \end{aligned}$$

for some polynomial $S(x)$. When $x = 1$, this takes the value $n!n^n$.

On the other hand,

$$\begin{aligned} D^n\left[\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}x^{kn}\right] &= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(kn)(kn-1)\cdots(kn-n+1)x^{kn-n} \\ &= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{(kn)!}{(kn-n)!}x^{kn-n} \\ &= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{kn}{n}n!x^{kn-n} \ . \end{aligned}$$

When $x = 1$, this takes the value $n!\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{kn}{n}$. The desired result follows.