PROBLEMS FOR MARCH 2007

Please send your solution to

Prof. E.J. Barbeau
Department of Mathematics
University of Toronto
Bahen Centre, Rom 6290
40 St. George St.
Toronto, ON M5S 2E4

no later than April 30, 2007. It is important that your complete mailing address and your email address appear on the front page. If you do not write your family name last, please underline it.

486. Determine all quintuplets (a, b, c, d, u) of nonzero integers for which

$$\frac{a}{b} = \frac{c}{d} = \frac{ab+u}{cd+u} \ .$$

487. ABC is an isosceles triangle with $\angle A = 100^\circ$ and $AB = AC$. The bisector of angle B meets AC in D. Show that $BD + AD = BC$.

488. A host is expecting a number of children, which is either 7 or 11. She has 77 marbles as gifts, and distributes them into n bags in such a way that whether 7 or 11 children come, each will receive a number of bags so that all 77 marbles will be shared equally among the children. What is the minimum value of n?

489. Suppose n is a positive integer not less than 2 and that $x_1 \geq x_2 \geq x_3 \geq \cdots \geq x_n \geq 0$,

$$\sum_{i=1}^n x_i \leq 400 \qquad \text{and} \qquad \sum_{i=1}^n x_i^2 \geq 10^4 \ .$$

Prove that $\sqrt{x_1} + \sqrt{x_2} \geq 10$. is it possible to have equality throughout? [*Bonus:* Formulate and prove a generalization.]

490. (a) Let a, b, c be real numbers. Prove that

$$\min \ [(a-b)^2, (c-a)^2, (a-b)^2] \leq \frac{1}{2}[a^2 + b^2 + c^2] \ .$$

(b) Does there exist a number k for which

$$\min \ [(a-b)^2, (a-c)^2, (a-d)^2, (b-c)^2, (b-d)^2, (c-d)^2] \leq k[a^2 + b^2 + c^2 + d^2]$$

for any real numbers a, b, c, d? If so, determine the smallest such k.
[*Bonus:* Determine if there is a generalization.]

491. Given that x and y are positive real numbers for which $x + y = 1$ and that m and n are positive integers exceeding 1, prove that

$$(1 - x^m)^n + (1 - y^n)^m > 1 \ .$$

492. The faces of a tetrahedron are formed by four congruent triangles. if α is the angle between a pair of opposite edges of the tetrahedron, show that

$$\cos\alpha = \frac{\sin(B - C)}{\sin(B + C)}$$

where B and C are the angles adjacent to one of these edges in a face of the tetrahedron.