Solutions for December

584. Let n be an integer exceeding 2 and suppose that $x_1, x_2, \cdots, x_n$ are real numbers for which $\sum_{i=1}^n x_i = 0$ and $\sum_{i=1}^n x_i^2 = n$. Prove that there are two numbers among the x_i whose product does not exceed -1.

Solution. We can supppose that the x_i are ordered in increasing sequence and that there is a positive integer k with $x_1 \leq x_2 \leq \cdots \leq x_k \leq 0 \leq x_{k+1} \leq \cdots \leq x_n$. Then, noting that $-x_1 \geq 0$, we have that

$$\sum_{i=1}^k x_i^2 \leq \sum_{i=1}^k x_1 x_i = -x_1(x_{k+1} + x_{k+2} + \cdots + x_n) \leq -(n-k)x_1 x_n$$

and

$$\sum_{i=k+1}^n x_i^2 \leq \sum_{i=k+1}^n x_n x_i = -x_n(x_1 + x_2 + \cdots + x_k) \leq -k x_1 x_n \ .$$

Finally, $n = \sum_{i=1}^n x_i^2 \leq -n x_1 x_n$; thus $x_1 x_n \leq -1$.

585. Calculate the number

$$a = \lfloor \sqrt{n-1} + \sqrt{n} + \sqrt{n+1} \rfloor^2 \ ,$$

where $\lfloor x \rfloor$ denotes the largest integer than does not exceed x and n is a positive integer exceeding 1.

Solution. It does not appear that there is a neat expression for this. One can obtain without too much trouble the inequality

$$3\sqrt{n-1} < \sqrt{n-1} + \sqrt{n} + \sqrt{n+1} < 3\sqrt{n} \ ,$$

from which we can find that when $k^2 + 1 \leq n \leq k^2 + (2k/3)$, then $\sqrt{a} = 3k$, when $k^2 + (2k/3) + (10/9) < n \leq k^2 + (4k/3) + (1/3)$, then $\sqrt{a} = 3k+1$, and when $k^2 + 4k + (13/9) < n \leq (k+1)^2$, then $\sqrt{a} = 3k+2$. However, this leaves the difficulty of getting the right expression for the gaps between the various ranges of n.

586. The function defined on the set $\mathbf{C^*}$ of all nonzero complex numbers satisfies the equation

$$f(z)f(iz) = z^2 \ ,$$

for all $z \in \mathbf{C}^*$. Prove that the function $f(z)$ is odd, *i,e.*, $f(-z) = -f(z)$ for all $z \in \mathbf{C}^*$. Give an example of a function that satisfies this condition.

Solution. Note that $f(z) \neq 0$ for all $x \in \mathbf{C}^*$. Replacing z by iz leads to $f(iz)f(-z) = -z^2$, from which we have that

$$f(z)f(iz) + f(iz)f(-z) = 0 \Longrightarrow f(z) + f(-z) = 0 \ .$$

Therefore the function is odd.

An example is given by $f(z) = (-1+i)z/\sqrt{2}$.

587. Solve the equation

$$\tan 2x \tan\left(2x + \frac{\pi}{3}\right) \tan\left(2x + \frac{2\pi}{3}\right) = \sqrt{3} \ .$$

Solution. Using the standard trigonometric identities for $\sin A \sin B$, $\cos A \cos B$, $\cos 2A$ and $\sin 2A$, we have that

$$\begin{aligned}\sqrt{3} &= \tan 2x \left(\frac{\sin(2x + (\pi/3))\sin(2x + (2\pi/3))}{\cos(2x + (\pi/3))\cos(2x + (2\pi/3))}\right) \\ &= \tan 2x \left(\frac{\cos(\pi/3) - \cos(4x + \pi)}{\cos(\pi/3) + \cos(4x + \pi)}\right) \\ &= \tan 2x \left(\frac{1 + 2\cos 4x}{1 - 2\cos 4x}\right) = \tan 2x \left(\frac{1 + 2(2\cos^2 2x - 1)}{1 - 2(1 - 2\sin^2 2x)}\right) \\ &= \left(\frac{\sin 2x}{\cos 2x}\right)\left(\frac{4\cos^2 2x - 1}{4\sin^2 2x - 1}\right) = \frac{2\sin 4x \cos 2x - \sin 2x}{2\sin 4x \sin 2x - \cos 2x} \\ &= \frac{\sin 6x + \sin 2x - \sin 2x}{\cos 2x - \cos 6x - \cos 2x} = \frac{\sin 6x}{-\cos 6x} = -\tan 6x \ .\end{aligned}$$

Therefore $x = -10^\circ + k \cdot 30^\circ$ for some integer k.

588. Let the function $f(x)$ be defined for $0 \leq x \leq \pi/3$ by

$$f(x) = \sec\left(\frac{\pi}{6} - x\right) + \sec\left(\frac{\pi}{6} + x\right) .$$

Determine the set of values (its image or range) assumed by the function.

Solution. Making use of the inequality $(1/a) + (1/b) \geq 2/\sqrt{ab}$ for $a, b > 0$, we find that

$$f(x) \geq \frac{2}{\sqrt{\cos((\pi/6) - x)\cos((\pi/6) + x)}} \geq \frac{2}{\sqrt{(1/4) + ((\cos 2x)/2)}} .$$

Since $0 \leq x \leq \pi/3$ implies that $-\frac{1}{2} \leq \cos 2x \leq 1$, it follows that

$$0 \leq \sqrt{\frac{1}{4} + \frac{\cos 2x}{2}} \leq \frac{\sqrt{3}}{2} ,$$

and

$$f(x) \geq \frac{4}{\sqrt{3}} .$$

Since $f(x)$ is continuous, $f(0) = 4/\sqrt{3}$ and $f(x)$ grows without bound when x approaches $\pi/3$, the image of f on $[0, \pi/3)$ is $[4/\sqrt{3}.\infty)$.

589. In a circle, A is a variable point and B and C are fixed points. The internal bisector of the angle BAC intersects the circle at D and the line BC at G; the external bisector of the angle BAC intersects the circle at E and the line BC at F. Find the locus of the intersection of the lines DF and EG.

Solution. Suppose without loss of generality that $AB > AC$. If M is the midpoint of BC, since $BG : GC = AB : AC$, $BG > GC$ so that G lies between M and C and A lies between E and F. Let P be the intersection of DF and EG.

Observe that D is the midpoint of the arc BC and that $AD \perp EF$. Therefore DA is an altitude of triangle DEF and DE is a diameter of the circle. Therefore DE must pass through M, and so $FM \perp DE$, *i.e.*, FM is an altitude of triangle DEF. The intersection of these two altitudes, G, is the orthocentre of triangle ABC and so $EG \perp DF$. Thus, $\angle EPD = 90^\circ$, so that P must lie on the given circle.

Conversely, let P be a point on the given circle. Wolog, we may assume that P lies between D, the midpoint of arc BC and C. Let DE be the diameter of the circle that right bisects BC. Suppose that DP produced intersects BC produced at F and that EF intersects the circle at A. This is the point A that produced the point P as described in the problem. Thus, the locus is indeed the given circle with the exception of the points B and C.

590. Let $SABC$ be a regular tetrahedron. The points M, N, P belong to the edges SA, SB and SC respectively such that $MN = NP = PM$. Prove that the planes MNP and ABC are parallel.

Solution. Let $|SM| = a$, $|SN| = b$ and $|SP| = c$. From the Law of Cosines, we have that $|MN|^2 = a^2 + b^2 - ab$, *etc.*, whence $a^2 + b^2 - ab = b^2 + c^2 - bc = c^2 + a^2 - ac = 0$. This implies that $a = b = c$ [prove it], so that $SM : SA = SN : SB = SP : SC$ and the result follows.