Solutions for July-August

556. Let x, y, z be positive real numbers for which $x + y + z = 4$. Prove the inequality

$$\frac{1}{2xy + xz + yz} + \frac{1}{xy + 2xz + yz} + \frac{1}{xy + xz + 2yz} \le \frac{1}{xyz} .$$

Solution. It is straightforward to establish for $a, b > 0$ that $(a + b)^{-1} \le \frac{1}{4}(a^{-1} + b^{-1})$. Therefore,

$$\begin{aligned}\frac{1}{2xy + xz + yz} &\le \frac{1}{4}\left(\frac{1}{xy + xz} + \frac{1}{xy + yz}\right) \le \frac{1}{4}\left[\frac{1}{4}\left(\frac{1}{xy} + \frac{1}{xz}\right) + \frac{1}{4}\left(\frac{1}{xy} + \frac{1}{yz}\right)\right] \\ &= \frac{1}{16}\left(\frac{2}{xy} + \frac{1}{xz} + \frac{1}{yz}\right) = \frac{1}{16}\left(\frac{2z + y + x}{xyz}\right) .\end{aligned}$$

Similarly,

$$\frac{1}{xy + 2xz + yz} \le \frac{1}{16}\left(\frac{z + 2y + x}{xyz}\right)$$

and

$$\frac{1}{xy + xz + 2yz} \le \frac{1}{16}\left(\frac{z + y + 2x}{xyz}\right) .$$

Adding the three inequalities yields that

$$\frac{1}{2xy + xz + yz} + \frac{1}{xy + 2xz + yz} + \frac{1}{xy + xz + 2yz} \le \frac{1}{16}\left(\frac{4x + 4y + 4z}{xyz}\right) = \frac{1}{xyz} .$$

Equality holds if and only if $x = y = z = 4/3$.

557. Suppose that the polynomial $f(x) = (1+x+x^2)^{1004}$ has the expansion $a_0+a_1x+a_2x^2+\cdots+a_{2008}x^{2008}$. Prove that $a_0 + a_2 + \cdots + a_{2008}$ is an odd integer.

Solution. Observe that

$$a_0 + a_2 + \cdots + a_{2008} = \frac{1}{2}(f(1) + f(-1)) = \frac{1}{2}(3^{1004} + 1) .$$

It remains to show that $3^{1004} + 1$ is congruent to 2 modulo 4.

558. Determine the sum

$$\sum_{m=0}^{n-1}\sum_{k=0}^{m}\binom{n}{k} .$$

Solution. Let $S_m = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{m}$. Then $S_0 + S_{n-1} = S_1 + S_{n-2} = \cdots = S_{n-1} + S_0 = 2^n$, so that $S = n2^{n-1}$.

Comment. In more detail,

$$\begin{aligned}S_k + S_{n-1-k} &= \left[\binom{n}{0} + \cdots + \binom{n}{k}\right] + \left[\binom{n}{0} + \cdots + \binom{n}{n-1-k}\right] \\ &= \left[\binom{n}{0} + \cdots + \binom{n}{k}\right] + \left[\binom{n}{n} + \cdots + \binom{n}{k+1}\right] = 2^n .\end{aligned}$$

559. Let ϵ be one of the roots of the equation $x^n = 1$, where n is a positive integer. Prove that, for any polynomial $f(x) = a_0 + a_x + \cdots + a_n x^n$ with real coefficients, the sum $\sum_{k=1}^{n} f(1/\epsilon^k)$ is real.

Solution. If $\epsilon = 1$, the result is clear. Let $\epsilon \neq 1$; we have that $\epsilon^n = 1$.

$$\begin{aligned}\sum_{k=1}^{n} f(1/\epsilon^k) &= \sum_{k=1}^{n}\sum_{j=0}^{n} a_j (1/\epsilon^k)^j = \sum_{k=1}^{n}\sum_{j=0}^{n} a_j (1/\epsilon^{jk}) \\ &= \sum_{j=0}^{n} a_j \sum_{k=1}^{n} (1/\epsilon^{jk}) = na_0 + \sum_{j=2}^{n-1} a_j (1/\epsilon^j)\left(\frac{1-\epsilon^{-jn}}{1-\epsilon^{-j}}\right) + na_n \\ &= na_0 + 0 + na_n = n(a_0 + a_n)\ .\end{aligned}$$

560. Suppose that the numbers $x_1, x_2, \cdots, x_n$ all satisfy $-1 \leq x_i \leq 1$ $(1 \leq i \leq n)$ and $x_1^3 + x_2^3 + \cdots + x_n^3 = 0$. Prove that

$$x_1 + x_2 + \cdots + x_n \leq \frac{n}{3}\ .$$

Solution. Since $-1 \leq x_i \leq 1$, for $1 \leq i \leq n$, there exists θ_i with $0 \leq \theta_i \leq \pi$ such that $x_i = \cos\theta_i$. Therefore

$$\begin{aligned}\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} \cos\theta_i &= \frac{1}{3}\left[4\sum_{i=1}^{n}\cos^3\theta_i - \sum_{i=1}^{n}\cos 3\theta_i\right] \\ &= -\frac{1}{3}\sum_{i=1}^{n}\cos 3\theta_i \leq \frac{n}{3}\ ,\end{aligned}$$

as desired.

561. Solve the equation

$$\left(\frac{1}{10}\right)^{\log_{(1/4)}(\sqrt[4]{x}-1)} - 4^{\log_{10}(\sqrt[4]{x}+5)} = 6\ ,$$

for $x \geq 1$.

Solution. Let $a = \log_{(1/4)}(\sqrt[4]{x} - 1)$ and $b = \log_{10}(\sqrt[4]{x} + 5)$. Then $(1/4)^a = \sqrt[4]{x} - 1$ and $10^b = \sqrt[4]{x} + 5$, whence $(1/4)^a + 1 = 10^b - 5$, or

$$\left(\frac{1}{4}\right)^a - 10^b = -6\ .$$

On the other hand, the given equation is

$$\left(\frac{1}{10}\right)^a - 4^b = 6\ .$$

Therefore

$$\left(\frac{1}{4}\right)^a - 4^b + \left(\frac{1}{10}\right)^a - 10^b = 0$$

which is equivalent to

$$(4^{-a} - 4^b) + (10^{-a} - 10^b) = 0\ .$$

The left side is less than 0 when $-a < b$ and greater than 0 when $-a > b$. Therefore $-a = b$ and so $10^b - 4^b = 6$. One solution of this is $b = 1$.

We show that this solution is unique. Observe that the function $f(x) = 6(1/10)^x + (4/10)^x$ decreases as x increases from 0 and takes the value 1 when $x = 1$. Since $f(x) = 1$ is equivalent to $6 = 10^x - 4^x$, we see that $x = 1$ is the only solution of the latter equation.

562. The circles $\mathfrak{C}$ and $\mathfrak{D}$ intersect at the two points A and B. A secant through A intersects $\mathfrak{C}$ at C and $\mathfrak{D}$ at D. On the segments CD, BC, BD, consider the respective points M, N, K for which $MN \| BD$ and $MK \| BC$. On the arc BC of the circle $\mathfrak{C}$ that does not contain A, choose E so that $EN \perp BC$, and on the arc BD of the circle $\mathfrak{D}$ that does not contain A, choose F so that $FK \perp BD$. Prove that angle EMF is right.

Solution. We have that $BN : NC = DM : MC = DK : KB$. Let G be the point of intersection of FK and $\mathfrak{D}$. Then $\angle BGD = \angle BAD = \angle BEC$. In triangle BGD and CEB, we have that $\angle BGD = \angle CEB$. Compare triangles BGD and CEB: $\angle BGD = \angle CEB$; GK and EN are respective altitudes; $DK : KB = BN : NC$. There is a similarilty transformation with factor $|DK|/|BN|$ that takes $B \rightarrow D$, $C \rightarrow B$, $N \rightarrow K$ and E to a point E' on the line KG. Since $\angle BGD = \angle CEB = \angle BE'D$, we must have $E' = G$. Thus triangles BGD and CEB are similar, whence $\angle EBC = \angle GDB = \angle GFB$. As a result, triangles BNE and FKB are similar.

Since $MNBK$ is a parallelogram, $\angle MNB = \angle MKB$. Thus $\angle MNE = \angle MKF$. Since $MN : KF = BK : KF = EN : NB = EN : MK$, triangles ENM and MKF are similar. Therefore $\angle NME = \angle KFM$. But $MN \perp KF$. Therefore $EM \perp FM$.