Solutions for March

535. Let the triangle ABC be isosceles with $AB = AC$. Suppose that its circumcentre is O, the D is the midpoint of side AB and that E is the centroid of triangle ACD. Prove that OE is perpendicular to CD.

Solution 1. Let F be the midpoint of AC, so that DF is a median of triangle ADC and so contains the point E. The centroid, G, of triangle ABC lies on the median CD as well as on the right bisector of BC. Since $DE \| BC$ and the circumcentre O of triangle ABC lies on the right bisector of BC, we have that $DE \perp AO$.

Let H be the midpoint of CD. Since FH is a midline of triangle ACD, $FH \| AD$. Since $DG = \frac{1}{3}CD = \frac{2}{3}DH$ and $DE = \frac{2}{3}DF$, $EG \| FH \| AB$. Since O lies on the right bisector of AB, $DO \perp EG$. Therefore, O is the intersection of two altitudes from D and G and so is the orthocentre of triangle DEG. Therefore $OE \perp CD$.

Solution 2. [N. Gurram] Place the configuration in a complex plane with O at 0 and A, B, C, respectively, at $2ai$, $-2b-2ci$, $2b-2ci$. Since $|OA| = |OB|$, $a^2 = b^2 + c^2$.

The point D is located at $-b + (a-c)i$ and E at

$$\frac{1}{3}[2ai + (-b + (a-c)i) + (2b - 2ci)] = \frac{1}{3}[b + 3(a-c)i] \ .$$

Note that $OE \perp CD$ if and only if $\frac{1}{3}[b+3(a-c)i]$ is i times a real multiple of $(2b-2ci)-(-b+(a-c)i) = 3b-(a+c)i$. Since

$$\begin{aligned} \frac{b+3(a-c)i}{3b-(a-c)i} &= \frac{[b+3(a-c)i][3b+(a+c)i]}{9b^2+(a-c)^2} \\ &= \frac{3[b^2-(a^2-c^2)] + [9b(a-c)+b(a+c)]i}{9b^2+(a-c)^2} \\ &= \frac{2b(5a-4c)i}{9b^2+(a-c)^2} \ , \end{aligned}$$

is pure imaginary, the result follows.

Solution 3. Assign coordinates to the points: $A \sim (2a, 2b)$, $B \sim (4a, 0)$, $C \sim (0, 0)$, and $O \sim (2a, k)$ where $4a^2 + k^2 = (2b-k)^2$ or $k = b - (a^2/b)$. Then $D \sim (3a, b)$ and $E \sim (5a/3, b)$. The slope of OE is $(-a^2/b)/(a/3) = -3a/b$ and the slope of CD is $b/3a$. Therefore $OE \perp CD$.

536. There are 21 cities, and several airlines are responsible for connections between them. Each airline serves five cities with flights both ways between all pairs of them. Two or more airlines may serve a given pair of cities. Every pair of cities is serviced by at least one direct return flight. What is the minimum number of airlines that would meet these conditions?

Solution 1. Since there are 210 pairs of cities and each airline serves 10 pairs, at least 21 airlines are required. In fact, we can get by with exactly 21 airlines. Label the cities from 0 to 20 inclusive, and let the kth airline service the set of five cities

$$\{k, k+2, k+7, k+8, k+11\}$$

where the numbers are taken modulo 21. Observe that the differences between two numbers of such sets for any airline cover all the numbers from 1 to 10. Given any two cities labelled i and j, the difference between the two labels (possibly adjusted modulo 21) is equal to some number between 1 and 10, and we can select a value of k for which the two labels appear in the cities services by the kth airline.

Solution 2. Suppose that there are m airlines, and that each airline maintains an office in each city that it serves. Then there are $5m$ offices. Consider any particular city: it is connected to four other cities by each airline that serves it, so that there must be at least $20/4 = 5$ offices in the city. Therefore, there are at least 5×21 offices in all the cities. Thus, $5k \geq 5 \times 21$ and so $k \geq 21$.

An example can be given as in the first solution.

537. Consider all 2×2 square arrays each of whose entries is either 0 or 1. A pair (A, B) of such arrays is *compatible* if there exists a 3×3 square array in which both A and B appear as 2×2 subarrays.

For example, the two arrays

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

are compatible, as both can be found in the array

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} .$$

Determine all pairs of 2×2 arrays that are not compatible.

Solution. Let a_{ij} and b_{ij} be the respective entries in the ith row and jth column of the arrays A and B, where $1 \leq i, j \leq 2$. If any of the following hold: $a_{11} = b_{22}$, $a_{21} = b_{12}$, $a_{12} = b_{21}$, $a_{22} = b_{11}$, then the arrays are compatible as they can be inserted into a 3×3 array overlapping at a corner. Therefore, if two arrays are not compatible, we must have that $b_{ij} = 1 - a_{ji}$ for each i and j.

Suppose that two matrices A and B related in this way have two unequal entries. Wolog, we may assume that $a_{11} = 0$ and $a_{12} = 1$. Then $b_{22} = 1$ and $b_{21} = 0$. Then the two matrices can be fitted into a 3×3 array with the bottom row of B coinciding with the top row of A. Hence, if A and B are not compatible, then each must have all of its entries the same. Therefore, the only noncompatible pairs (A, B) have one matrix containing only 1s and the other only 0s.

538. In the convex quadrilateral $ABCD$, the diagonals AC and BD are perpendicular and the opposite sides AB and DC are not parallel. Suppose that the point P, where the right bisectors of AB and DC meet, is inside $ABCD$. Prove that $ABCD$ is a cyclic quadrilateral if and only if the triangles ABP and CDP have the same area.

Solution 1. [N. Gurram] Let AC and BD intersect at E. Let R and S be the respective feet of the perpendiculars from P to AC and BD. Observe that $PRES$ is a rectangle, so that $|RE| = |PS|$ and $|SE| = |PR|$. T here are essentially two cases to consider, according as P lies in triangle AEB or triangle AED.

First, suppose that P lies in triangle AEB. Then

$$\begin{aligned} [ABP] &= [ABE] - [AEP] - [BEP] \\ &= \frac{1}{2}[|AE||BE| - |AE||ES| - |BE||ER|] \\ &= \frac{1}{2}[(|AE| - |ER|)(|BE| - |ES|) - |ER||ES|] \\ &= \frac{1}{2}[|AR||BS| - |ER||ES|] \ . \end{aligned}$$

Likewise,

$$\begin{aligned} [CDP] &= [CDE] + [CEP] + [DEP] \\ &= \frac{1}{2}[|CE||DE| + |CE||ES| + |DE||ER|] \\ &= \frac{1}{2}[(|CE| + |ER|)(|DE| + |ES|) - |ER||ES|] \\ &= \frac{1}{2}[|CR||DS| - |ER||ES|] \ . \end{aligned}$$

Secondly, suppose that P lies in triangle AED. Then

$$\begin{aligned}[ABP] &= [ABE] + [AEP] - [BEP] \\ &= \frac{1}{2}[|AE||BE| + |AE||ES| - |BE||ER|] \\ &= \frac{1}{2}[(|AE| - |ER|)(|BE| + |ES|) + |ER||ES|] \\ &= \frac{1}{2}[|AR||BS| + |ER||ES|] \ .\end{aligned}$$

Likewise,

$$\begin{aligned}[CDP] &= [CDE] - [CEP] + [DEP] \\ &= \frac{1}{2}[|CE||DE| - |CE||ES| + |DE||ER|] \\ &= \frac{1}{2}[(|CE| + |ER|)(|DE| - |ES|) + |ER||ES|] \\ &= \frac{1}{2}[|CR||DS| + |ER||ES|] \ .\end{aligned}$$

In either case, we find that

$$[ABP] - [CDP] = \frac{1}{2}(|AR||BS| - |CR||DS| \ .$$

Suppose that $ABCD$ is concyclic. Then P is the centre of the circumcircle of $ABCD$, and hence of each of the triangle ABC and ABD. Therefore, R is the midpoint of AC and S the midpoint of $BD/$ Therefore, $|AR| = |CR|$ and $|BS| = |DS|$, so that $[ABP] = [CDP]$.

On the other hand, suppose that $ABCD$ is not concyclic. Then, wolog, we may suppose that $|AP| = |BP| > |CP| = |DP|$. By looking at right triangles, we see that $|AR| > |CR|$ and $|BS| > |DS|$, so that $[ABP] > [CDP]$. The result follows.

Solution 2. [J. Zung] We first establish Brahmagupta's theorem: *Suppose that $ABCD$ is a concyclic quadrilateral and that AC and BD intersect at right angles at E. Let Q be the point on CD for which $EQ \perp CD$, and let QE produced meet AB at M. Then M is the midpoint of AB.*

To prove this, note that

$$\begin{aligned}\angle MEB = \angle DEQ = 90^\circ - \angle EDQ = 90^\circ - \angle EDC \\ = \angle DCA = \angle DBA = \angle EBM \ ,\end{aligned}$$

whence $MB = ME$. Similarly, $MA = ME$. ♠

In the problem, let $ABCD$ be concyclic with circumcentre P, and M and N be the respective midpoints of AB and CD. Since P is the circumcentre of $ABCD$, we have that $PN \perp CD$, so that $PN \| ME$ by Brahmagupta's theorem. Similarly, $PM \| NE$ so that $PMNE$ is a paralleloram.

Therefore,

$$[CDP] = |ND||PN| = |NE||PN| = |PM||ME| = |PM||AM| = [ABP] \ .$$

[Z.Q. Liu] Suppose that the respective midpoints of AB and CD are M and N, that AC and BD intersect at E, that the right bisectors of AB and CD meet at P, and that AB and DC produced meet at K. Observe that, because of the right triangle ABE and CDE, $MA = MB = ME$ and $NC = ND = NE$.

Suppose that $[ABP] = [CDP]$. Then

$$|AM||MP| = |DN||NP| \Longrightarrow |ME||MP| = |NE|NP| \Longrightarrow ME : NE = NP : MP \ .$$

Also

$$\begin{aligned} \angle MEN &= \angle MEA + \angle NED + 90^\circ = \angle MAE + \angle NDE + 90^\circ \\ &= \angle KAD + \angle KDA = 180^\circ - \angle MKN = \angle MPN \ , \end{aligned}$$

since $\angle KMP = \angle KNP = 90^\circ$. Therefore triangles MEN and MPN are similar. But as their common side MN corresponds in the similarity, the two triangles are congruent and so $MENP$ is a parallelogram.

Suppose that ME produced meets CD at R. Then $MR \perp CD$ and

$$\angle BDC = \angle EDR = \angle NER = \angle AEM = \angle MAE = \angle BAC \ ,$$

from which we conclude that $ABCD$ is concyclic.

Solution 3. (part) [T. Tang] As before, let the diagonals intersect at E, the right bisectors of AB and CD intersect at P and M and N be the respective midpoints of AB and CD. Suppose that $ABCD$ is concyclic. Then P is the circumcentre,

$$\begin{aligned} \angle NCP &= 90^\circ - \angle NPC = 90^\circ - \frac{1}{2}\angle DPC \\ &= 90^\circ - \angle DBC = 90^\circ - \angle EBC \\ &= \angle BCE = \angle BCA = \frac{1}{2}\angle APB = \angle MPB \ , \end{aligned}$$

and

$$\angle PCD = 90^\circ - \angle NCP = 90^\circ - \angle MPB = \angle MBP \ .$$

As $PB = PC$, triangle PMB and CNP are congruent, so that $[APB] = 2[PMB] = 2[CNP] = [CPD]$.

539. Determine the maximum value of the expression

$$\frac{xy + 2yz + zw}{x^2 + y^2 + z^2 + w^2}$$

over all quartuple of real numbers not all zero.

Solution 1. Observe that

$$\begin{aligned} 0 &\le [x - (\sqrt{2} - 1)y]^2 = x^2 + (3 - 2\sqrt{2})y^2 - 2(\sqrt{2} - 1)xy \ , \\ 0 &\le [w - (\sqrt{2} - 1)z]^2 = w^2 + (3 - 2\sqrt{2})w^2 - 2(\sqrt{2} - 1)zw \ , \end{aligned}$$

and

$$0 \le 2(\sqrt{2} - 1)(y - z)^2 = 2(\sqrt{2} - 1)y^2 + 2(\sqrt{2} - 1)z^2 - 4(\sqrt{2} - 1)yz \ ,$$

with equality if and only if $x = (\sqrt{2} - 1)y = (\sqrt{2} - 1)z = w$. Adding the inequalities yields

$$2(\sqrt{2} - 1)(xy + 2yz + zw) \le x^2 + y^2 + z^2 + w^2 \ .$$

Therefore, the maximum value of the expression is $[2(\sqrt{2}-1)]^{-1} = \frac{1}{2}(\sqrt{2}+1)$, and this maximum is assumed, for example, when $(x, y, z, w) = (\sqrt{2} - 1, 1, 1, \sqrt{2} - 1)$.

Solution 2. Since the expression is homogeneous of degree 0, we may wolog assume that $x^2+y^2+w^2+z^2 = 1$. Select θ so that $0 \le \theta \le \pi/2$ and $y^2 + z^2 = \sin^2\theta$ and $x^2 + w^2 = \cos^2\theta$. Then $2yz \le \sin^2\theta$ and, by the Cauchy-Schwarz Inequality, $xy + zw \le \sin\theta\cos\theta$. Therefore

$$\begin{aligned} xy + 2yz + zw &\le \sin^2\theta + \sin\theta\cos\theta \\ &= \frac{1}{2}[1 - \cos 2\theta + \sin 2\theta] \\ &= \frac{1}{2}\left[1 + \sqrt{2}\sin\left(2\theta - \frac{\pi}{4}\right)\right] \\ &\le \frac{1}{2}[1 + \sqrt{2}] \ , \end{aligned}$$

with equality if and only if $\theta = \frac{3\pi}{8}$.

Solution 3. [H. Spink] Let u and v be nonnegative real numbers for which $2u^2 = x^2+w^2$ and $2v^2 = y^2+z^2$. Then $2yz \leq 2v^2$, $xy + zw \leq 2uv$ (by the Cauchy-Schwarz Inequality) and $x^2 + y^2 + z^2 + w^2 = 2(u^2 + v^2)$. The given expression is not greater than $(v^2 + uv)/(u^2 + v^2)$. Equality occurs when $x = w$ and $y = z$. This vanishes when $v = 0$, When $v \neq 0$, we can write it as

$$f(w) \equiv \frac{1+w}{1+w^2}$$

where $w = u/v$. Thus, it suffices to determine the maximum of this last expression over positive values of w.

$f(w)$ assumes the positive real value λ if and only if the equation $f(w) = \lambda$ is solvable. This equation can be rewritten as

$$\begin{aligned} 0 &= \lambda w^2 - w + (\lambda - 1) \\ &= \frac{1}{4\lambda}[4\lambda^2 w^2 - 4\lambda w + 4\lambda(\lambda - 1)] \\ &= \frac{1}{4\lambda}[(2\lambda w - 1)^2 + (2\lambda - 1)^2 - 2] \ . \end{aligned}$$

The equation is solvable if and only if

$$(2\lambda - 1)^2 \leq 2 \Longleftrightarrow \lambda \leq \frac{\sqrt{2}+)}{2} \ .$$

The value of w that yields this value of λ is

$$\frac{1}{2\lambda} = \frac{1}{\sqrt{2}+1} = \sqrt{2} - 1 \ .$$

The expression takes its maximum value of $\frac{1}{2}(\sqrt{2}+1)$ when $(x, y, z, w) = (\sqrt{2}-1, 1, 1, \sqrt{2}-1)$.

Solution 4. [J. Zung] Let the expression to be maximized by u and set $x = a + b$, $y = a - b$, $z = c + d$, $w = c - d$. Then

$$u = \frac{ac + c^2 + bd - d^2}{a^2 + b^2 + c^2 + d^2} \ .$$

When q and s are positive, then $(p+r)/(q+s)$ lies between p/q and r/s, with equality if and only if $p/q = r/s$. Applying this to u, we see that it lies between

$$\frac{bd - d^2}{b^2 + d^2} \qquad \text{and} \qquad \frac{ac + c^2}{a^2 + c^2} \ .$$

The term on the left, being no greater than, $(bd + d^2)/(b^2 + d^2)$ is less than the maximum value over all (a, c) of the term on the right. So we maximize the function of a and c. Since it vanishes when $c = 0$ and clearly takes positive values, we may assume $c \neq 0$ and that $w = a/c$. Thus, we maximize $(1 + w)/(1 + w^2)$. This can be done as in Solution 3 to obtain the maximum value $\frac{1}{2}(\sqrt{2}+1)$.

However, we are not quite done. To ensure that u can assume this value, it seems that we need to find (b, d) so that $(bd - d^2)/(b^2 + d^2)$ equals this maximum value of $(ac + c^2)/(a^2 + c^2)$. But there is a way out: u is equal to the maximum when $b = d = 0$, and this occurs when $x = w$ and $y = z$, leading to the solution given previously.

Solution 5. [P. Wen] We are looking for the smallest value of u for which

$$\frac{xy + 2yz + zw}{x^2 + y^2 + z^2 + w^2} \leq u$$

for all reals x, y, z, w, not all vanishing. Since $|xy + 2yz + zw| \leq |x||y| + 2|y||z| + |z||w|$, it is enough to consider only nonnegative values of the variables. Since the left side takes the value 1 when $x = y = z = w$, and eligible u satisfies $u \geq 1$.

The inequality can be rewritten

$$\begin{aligned}
0 &\leq u(x^2 + y^2 + z^2 + w^2) - (xy + 2yz + zw) \\
&= (y - z)^2 + (u - 1)(y^2 + z^2) + u(x^2 + w^2) - xy - zw \\
&= (y - z)^2 + [\sqrt{u-1}y - \sqrt{u}x]^2 + [\sqrt{u-1}z - \sqrt{w}w]^2 + (\sqrt{(u-1)u} - 1)(xy + zw) \ .
\end{aligned}$$

This is to hold for all $x, y, z, w \geq 0$. When $y = z$ and $\sqrt{u}x = \sqrt{u-1}y$, the first three terms on the right vanish; for the fourth to be nonnegative, we require that

$$2\sqrt{(u-1)u} - 1 \geq 0 \quad \Longleftrightarrow 1 \leq 4(u-1)u \quad \Longleftrightarrow \quad 2 \leq (2u-1)^2 \quad 2u - 1 \geq \sqrt{2} \ .$$

Thus $u \geq \frac{1}{2}(\sqrt{2} + 1)$.

When $u = \frac{1}{2}(\sqrt{2}+1)$, $\sqrt{(u-1)/u} = \sqrt{2} - 1$, and we find that the expression in the problem assumes the value $\frac{1}{2}(\sqrt{2}+1)$ when $(x, y, z, w) = (\sqrt{2} - 1, 1, 1, \sqrt{2} - 1)$. Thus, the maximum value is $\frac{1}{2}(\sqrt{2}+1)$.

540. Suppose that, if all planar cross-sections of a bounded solid figure are circles, then the solid figure must be a sphere.

Solution. Since the solid figure is bounded, there exists two point A and B whose distance, r, apart is maximum. Let σ be any plane that passes through the segment AB. It intersects the solid figure in a circle, and no two points on this circle can be further than r apart. Therefore, AB is a diameter of this circle, and the solid figure is the solid of revolution of this circle about the segment AB.

541. Prove that the equation
$$x_1^{x_1} + x_2^{x_2} + \cdots + x_k^{x_k} = x_{k+1}^{x_{k+1}}$$
has no solution for which x_1, x_2, $\cdots$, x_k, x_{k+1} are all distinct nonzero integers.

Solution. Consider a sum of the following type:

$$\sum_{r=2}^{\infty} \epsilon_r r^{-r} = \epsilon_2 \frac{1}{2^2} + \epsilon_3 \frac{1}{3^3} + \epsilon_4 \frac{1}{4^4} + \epsilon_5 \frac{1}{5^5} + \cdots \ ,$$

where each ϵ_r is one of the numbers -1, 0, 1 and at most finitely many ϵ_r are nonzero. Since

$$\frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \cdots < \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots \leq \frac{1}{2} \ ,$$

it follows that each sum must exceed $-1/2$ and be less than $1/2$. Furthermore, since, for each index $s \geq 2$,

$$\begin{aligned}
&\frac{1}{(s+1)^{(s+1)}} + \frac{1}{(s+2)^{(s+2)}} + \frac{1}{(s+3)^{(s+3)}} + \frac{1}{(s+4)^{(s+4)}} + \cdots \\
&< \frac{1}{(s+1)^{(s+1)}} + \frac{1}{(s+1)^{(s+2)}} + \frac{1}{(s+1)^{(s+3)}} + \frac{1}{(s+1)^{(s+4)}} + \cdots \leq \frac{1}{s(s+1)^s} < \frac{1}{s^s} \ ,
\end{aligned}$$

it follows that for each sum, the absolute value of the first nonzero term exceeds the absolute value of the sum of the remaining terms, so that no sum can vanish. Therefore, all sums of the prescribed type are nonintegral rationals between $-1/2$ and $1/2$.

Suppose that there is a solution in integers to the equation of the problem; wolog, we may take $x_1 < x_2 < x_3 < \cdots < x_k$. If $x_1 \leq -2$, then by the result of the previous paragraph, the sum of the terms of the left side is not an integer. Therefore x_{k+1}^{k+1} is not an integer, so that $x_{k+1} \leq -2$. Shifting this term to the left side, we get a sum equal to zero consisting of two parts, a sum of the type $\sum_{r=2}^{\infty} \epsilon_r r^{-r}$ which is a noninteger and a sum of integer terms $x_i^{x_i}$ corresponding to any terms $x_i \geq -1$. This is impossible. Therefore, for all i, $x_i \geq -1$.

There is no solution in the case that $k = 1$. When $k \geq 2$, we must have that $x_1 \geq -1$, $x_k \geq 2$ and $x_k \geq k - 1$. Therefore $x_{k+1}^{k+1} \geq x_k^{x_k} - 1$, whence $x_{k+1} > x_k$. Also

$$x_1^{x_1} + x_2^{x_2} + \cdots + x_k^{x_k} < kx_k^{x_k} \leq (x_k + 1)x_k^{x_k} < (x_k + 1)^{(x_k+1)} \leq x_{k+1}^{x_{k+1}}.$$

It follows that the equation is not solvable for distinct integers values of the x_i.