Solutions for May

549. The set E consists of 37 two-digit natural numbers, none of them a multiple of 10. Prove that, among the elements of E, we can find at least five numbers, such that any two of them have different tens digits and different units digits.

Solution. Call a set of nine numbers with the same tens digit a *decade*. By the Pigeon-Hole Principle, there is at least one decade with at least five numbers of E in it; wolog, let it be the tens decade. There are at least 28 numbers in E that are not in the tens decade; one of the remaining decades, say the twenties, must have at least four members of E. There are at least 19 members of E that are not in the tens or twenties decade; at least one of the remaining decades, say the thirties, has at least three members of E. Similarly, a fourth decade, say the forties, has at least two members and a fifth decade, say the fifties, has at least one member.

We can select the five element subset of E working back from the fifth decade. Select any number from the fifties, a number from the forties with a different tens digit, a number from the thirties with a tens digit differing from the two already determined, a number from the twenties with a tens digit differing from the three already determined and finally a number from the tens with a fifth tens digit. This will serve the purpose.

550. The functions $f(x)$ and $g(x)$ are defined by the equations: $f(x) = 2x^2 + 2x - 4$ and $g(x) = x^2 - x + 2$.

(a) Find all real numbers x for which $f(x)/g(x)$ is a natural number.

(b) Find the solutions of the inequality

$$\sqrt{f(x)} + \sqrt{g(x)} \geq \sqrt{2} \ .$$

Solution. (a) We have that

$$\frac{f(x)}{g(x)} = \frac{2x^2 + 2x - 4}{x^2 - x + 2} = \frac{2(x+2)(x-1)}{x^2 - x + 2} = 2 + \frac{4(x-2)}{x^2 - x + 2} \ .$$

Observe that $x^2 - x + 2 = (x - \frac{1}{2})^2 + \frac{7}{4} > 0$.

Since $x^2 - 5x + 10 = (x - \frac{5}{2})^2 + \frac{15}{4} > 0$, $4x - 8 < x^2 - x + 2$, so that $4(x-2)/(x^2 - x + 2) < 1$. Hence $f(x)/g(x)$ cannot take integer values exceeding 2.

$$f(x)/g(x) = 2 \Longleftrightarrow x = 2 \ ;$$

$$f(x)/g(x) = 1 \Longleftrightarrow 4x - 8 = -(x^2 - x + 2) \Longleftrightarrow x^2 + 3x - 6 = 0 \ .$$

Therefore, $f(x)/g(x)$ is a natural number if and only if $x = 2$ or $x = \frac{1}{2}(-3 \pm \sqrt{3})$.

Comment. It is not too hard to find all values of x for which $f(x)/g(x)$ assumes integer values. Note that, if $x \geq 2$ or $x \leq -5$, then $x^2 + 3x - 6 > 0$, so that $4x - 8 > -(x^2 - x + 2)$ and $4(x-2)/(x^2 - x + 2) > -1$. Thus, if $f(x)/g(x)$ assumes integer values, then $|x| \leq 5$ and

$$\left| \frac{4(x-2)}{x^2 - x + 2} \right| \leq \frac{12}{3/2} = 8 \ .$$

Thus, $f(x)/g(x)$ can takes only integer values between -6 and 2, and we can check each case.

(b) We require that $f(x) \geq 0$, so that $x \leq -2$ or $x \geq 1$. The functions $f(x) = 2(x+2)(x-1)$ and $g(x) = x(x-1) + 2$ are both decreasing on $(-\infty, -2]$ and increasing on $[1, +\infty)$. Since $\sqrt{f(-2)} + \sqrt{g(-2)} = 0 + 2\sqrt{2} > 2$, $\sqrt{f(1)} + \sqrt{g(1)} = \sqrt{2}$, the inequality is satisfied on the set $(-\infty, -2] \cup [1, +\infty)$.

Note. There was an error in the statement of (b), where $\sqrt{2}$ was given as 2. In this case, the inequality is satisfied on the set $(\infty, -2] \cup [\alpha, \infty)$, where α lies between 1 and 2, and satisfies the equation $\sqrt{f(\alpha)} + \sqrt{g(\alpha)} = 2$.

551. The numbers 1, 2, 3 and 4 are written on the circumference of a circle, in this order. Alice and Bob play the following game: On each turn, Alice adds 1 to two adjacent numbers, while Bob switches the places of two adjacent numbers. Alice wins the game, if after her turn, all numbers on the circle are equal. Does Bob have a strategy to prevent Alice from winning the game? Justify your answer.

Solution. Bob can prevent Alice from winning the game whenever Alice has the first move. The configuration of numbers begins with even and odd numbers alternating; Bob's strategy is to always present Alice with this situation. Then, whatever Alice does, she must leave two odd and two even numbers and therefore at least two distinct numbers.

To do this, Bob must switch whatever pair of numbers Alice selects to add 1 to. Alice's move changes the parity of the numbers in the two positions and Bob's move switches the parity back to what it was before.

552. Two real nonnegative numbers a and b satisfy the inequality $ab \geq a^3 + b^3$. Prove that $a + b \leq 1$.

Solution 1. Note that $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ and that $a^2 - ab + b^2$ is always positive. Then

$$\begin{aligned} 1 - (a + b) &= 1 - \frac{a^3 + b^3}{a^2 - ab + b^2} \geq 1 - \frac{ab}{a^2 - ab + b^2} \\ &= \frac{a^2 - 2ab + b^2}{a^2 - ab + b^2} = \frac{(a - b)^2}{a^2 - ab + b^2} \geq 0 \ , \end{aligned}$$

from which the desired result follows.

Solution 2. We note two inequalities: (1) $(a + b)^2 \geq 4ab$ and (2) $4(a^3 + b^3) \geq (a + b)^3$. The first is a consequence of the arithmetic-geometric means inequality, while the second can be obtained either from $(a + b)(a - b)^2 \geq 0$ or from the power-mean inequality $[\frac{1}{2}(a^3 + b^3)]^{1/3} \geq \frac{1}{2}(a + b)$. It follows that

$$(a + b)^2 \geq 4ab \geq 4(a^3 + b^3) \geq (a + b)^3 \ ,$$

from which the result is obtained.

553. The convex quadrilateral $ABCD$ is concyclic with side lengths $|AB| = 4$, $|BC| = 3$, $|CD| = 2$ and $|DA| = 1$. What is the length of the radius of the circumcircle of $ABCD$? Provide an exact value of the answer.

Solution. Let $\alpha = \angle DAB$ and $\beta = \angle ABC$, so that $\angle BCD = 180° - \alpha$ and $\angle CDA = 180° - \beta$. Then, by the Law of Cosines,

$$1 + 16 - 8\cos\alpha = |BD|^2 = 9 + 4 + 12\cos\alpha \ ,$$

whence $\cos\alpha = 1/5$ and $|BD| = \sqrt{77/5}$. The circumradius R of $ABCD$ satisfies $2R\sin\alpha = |BD|$, whence

$$R = \frac{\sqrt{77/5}}{2\sqrt{24/25}} = \frac{\sqrt{5 \times 7 \times 11}}{4\sqrt{6}} = \frac{\sqrt{385}}{4\sqrt{6}} \ .$$

As a check, we can find that

$$16 + 9 - 24\cos\beta = |AC|^2 = 4 + 1 + 4\cos\beta \ ,$$

whence $\cos\beta = 5/7$ and $|AC| = \sqrt{\frac{55}{7}}$. Thus, $2R\sin\beta = |AC|$, so that

$$R = \frac{\sqrt{55/7}}{2\sqrt{24/49}} = \frac{\sqrt{5\times 7\times 11}}{4\sqrt{6}} = \frac{\sqrt{385}}{4\sqrt{6}} \ .$$

554. Determine all real pairs (x, y) that satisfy the system of equations:

$$3\sqrt[3]{x^2y^5} = 4(y^2 - x^2)$$

$$5\sqrt[3]{x^4y} = y^2 + x^2 \ .$$

Solution. Multiply the two equations to obtain

$$15x^2y^2 = 4(y^4 - x^4) \Leftrightarrow 0 = 4x^4 + 15x^2y^2 - 4y^4 = (4x^2 - y^2)(x^2 + 4y^2)$$
$$\Leftrightarrow y^2 = 4x^2 \Leftrightarrow y = \pm 2x \ .$$

Substituting $y = 2x$ into the first equation yields that

$$3\sqrt[3]{2^5x^2x^5} = 12x^2 \Longrightarrow 2^5\times 3^3\times x^7 = 2^6\times 3^3\times x^6 \Longrightarrow x = 0 \text{ or } x = 2 \ .$$

Similarly, substituting $y = -2x$ into the first equation yields the additional solution $x = -2$. There are three solutions to the system, namely, $(x, y) = (0, 0), (2, 4), (-2, 4)$. All check out.

555. Let ABC be a triangle, all of whose angles do not exceed $90°$. The points K on side AB, M on side AC and N on side BC are such that $KM \perp AC$ and $KN \perp BC$. Prove that the area $[ABC]$ of triangle ABC is at least 4 times as great as the area $[KMN]$ of triangle KMN, *i.e.*, $[ABC] \geq 4[KMN]$. When does equality hold?

Solution 1. Let $b = |AC|$, $a = |BC|$, $m = |KM|$, $n = |KN|$ and $\theta = \angle ACB$, so that $\angle MKN = 180° - \theta$. Then, by the arithmetic-geometric means inequality,

$$[ABC] = [AKC] + [AKB] = \frac{1}{2}(bm + an) \geq \sqrt{abmn} \ .$$

Therefore

$$\begin{aligned}[ABC]^2 &\geq abmn \geq abmn\sin^2\theta \\ &\geq (ab\sin\theta)(mn\sin(180° - \theta)) \\ &= 2[ABC]\cdot 2[KMN] \ ,\end{aligned}$$

whence $[ABC] \geq 4[KMN]$, as desired.

Equality holds if and only if $bm = an$ and $\sin\theta = 1$, if and only if $a : b = m : n$ and $\theta = 90°$, if and only if triangle ABC is right and similar to triangles AKM and KBN. In this case, $KN\|AC$ and $KM\|BC$ and the linear dimensions of KMN are half those of CAB; thus, $AC = 2NK$, $BC = 2MK$ and K is the midpoint of AB.

Solution 2. Let the angles at A, B and C in triangle ABC be respectively α, β, γ and the sides of this triangle be, conventionally, a, b, c. Suppose that $m = |KM|$, $n = |KN|$, and $x = |AK|$, so that $m = x\sin\alpha$ and $n = (c - x)\sin\beta$.

Then $[ABC] = \frac{1}{2}ab\sin\gamma$ and

$$[KMN] = \frac{1}{2}mn\sin(180° - \gamma) = \frac{1}{2}x(c - x)\sin\alpha\sin\beta\sin\gamma \ .$$

Thus

$$\frac{[KMN]}{[ABC]} = \frac{x(c-x)\sin\alpha\sin\beta}{ab} = x(c-x)\left(\frac{\sin\alpha}{a}\right)\left(\frac{\sin\beta}{b}\right)$$
$$= x(c-x)\left(\frac{1}{2R}\right)\left(\frac{1}{2R}\right) = \frac{x(c-x)}{4R^2}\ ,$$

where R is the circumradius of triangle ABC. By the Arithmetic-Geometric Means Inequality, $x(c-x) \leq (\frac{1}{2}[x+(c-x)])^2 = c^2/4$, so that

$$\frac{[KMN]}{[ABC]} = \frac{x(c-x)}{4R^2} \leq \left(\frac{c^2}{4}\right)\left(\frac{1}{4R^2}\right) = \frac{1}{4}\left(\frac{c}{2R}\right)^2$$
$$= \frac{1}{4}\sin^2\gamma \leq \frac{1}{4}\ ,$$

as desired. Equality holds if and only if $x = c - x$ and $\sin\gamma = 1$, *i.e.*, when ABC has a right angle at C and K is the midpoint of AB.