Solutions for April

612. $ABCD$ is a rectangle for which $AB > AD$. A rotation with centre A takes B to a point B' on CD; it takes C to C' and D to D'. Let P be the point of intersection of the lines CD and $C'D'$. Prove that $CB' = DP$.

Solution 1. [N. Lvov; K. Zhou] Since $\angle CB'P = 90° - \angle DB'A = \angle DAB'$ and $AD = BC = B'C'$, triangles $AB'D$ and $B'PC$ are congruent (ASA). Therefore

$$\begin{aligned} DP &= B'P - B'D = AB' - B'D \\ &= AB - B'D = CD - B'D = CB' \ . \end{aligned}$$

Solution 2. Let the respective lengths of AB and BC be a and b respectively, and suppose that the rotation about A is through the angle 2α. Then $\angle CBB' = \alpha$ and we find that

$$\begin{aligned} a &= b(\tan\alpha + \cot 2\alpha) \\ &= b\left(\frac{\sin\alpha}{\cos\alpha} + \frac{\cos 2\alpha}{\sin 2\alpha}\right) \\ &= b\left(\frac{2\sin^2\alpha + 1 - 2\sin^2\alpha}{\sin 2\alpha}\right) \\ &= b\left(\frac{1}{\sin 2\alpha}\right) \ . \end{aligned}$$

Since $|B'C'| = b$ and $\angle C'B'P = 90° - 2\alpha$, then $\angle B'PC' = 2\alpha$. Thus $\sin 2\alpha = |B'C'|/|B'P|$, so that $|B'P| = b/\sin 2\alpha = a = |CD|$. The result follows.

Solution 3. [A. Dhawan] The circle with centre A and radius $|AD|$ passes though D and D'; the tangent through P are PD and PD' and so

$$\angle DAP = \frac{1}{2}\angle D'AD = \frac{1}{2}\angle B'AB \ .$$

Also, we have that

$$\angle B'BC = 90° - \angle B'BA = \frac{1}{2}(180° - \angle B'BA - \angle BB'A) = \frac{1}{2}\angle B'AB \ ,$$

so that $\angle PAD = \angle B'BC$. Since also $\angle PDA = 90° = \angle B'CB$ and $DA = CB$, triangles PDA and $B'CB$ are congruent (ASA). Therefore $PD = B'C$.

613. Let ABC be a triangle and suppose that

$$\tan\frac{A}{2} = \frac{p}{u} \qquad \tan\frac{B}{2} = \frac{q}{v} \qquad \tan\frac{C}{2} = \frac{r}{w} \ ,$$

where p, q, r, u, v, w are positive integers and each fraction is written in lowest terms.

(a) Verify that $pqw + pvr + uqr = uvw$.

(b) Let f be the greatest common divisor of the pair $(vw - qr, qw + vr)$, g be the greatest common divisor of the pair $(uw-pr, pw+ur)$, and h be the greatest common divisor of the pair $(uv-pq, pv+qu)$. Prove that

$$fp = vw - qr \qquad fu = qw + vr$$

$$gq = uw - pr \qquad gv = pw + ur$$

$$hr = uv - pq \qquad\qquad hw = pv + qu \ .$$

(c) Prove that the sides of the triangle ABC are proportional to $fpu : gqv : hrw$.

Solution 1. Since $A/2$ and $B/2 + C/2$ are complementary, $\cot(A/2) = \tan(B/2 + C/2)$, whence

$$\frac{u}{p} = \frac{qw + vr}{vw - qr} \ .$$

Parts (a) and (b) follow immediately.

The sides of the triangle are proportional to $\sin A : \sin B : \sin C$. Now

$$\sin A = \frac{2\tan\frac{A}{2}}{\sec^2\frac{A}{2}} = \frac{2pu}{p^2 + u^2} = \frac{2fpu}{f(p^2 + u^2)} \ ;$$

$$\sin B = \frac{2\tan\frac{B}{2}}{\sec^2\frac{B}{2}} = \frac{2qv}{q^2 + v^2} = \frac{2gqv}{g(q^2 + v^2)} \ ;$$

$$\sin C = \frac{2\tan\frac{C}{2}}{\sec^2\frac{C}{2}} = \frac{2rw}{r^2 + w^2} = \frac{2hrw}{h(r^2 + w^2)} \ .$$

From (b), we have that

$$f^2(p^2 + u^2) = (q^2 + v^2)(r^2 + w^2)$$

so that

$$f^2(p^2 + u^2)^2 = (p^2 + u^2)(q^2 + v^2)(r^2 + w^2) \ .$$

Similar equations hold for g and h. We find that

$$f(p^2 + u^2) = g(q^2 + v^2) = h(r^2 + w^2) \ .$$

Hence $\sin A : \sin B : \sin C = fpu : gqv : hrw$ as desired.

Solution 2. (a) and (b) can be obtained as above. For (c), let x, y, z be the respective distances from A, B, C to the adjacent tangency points of the incircle of triangle ABC. Then $\tan A/2 = r/x$, $\tan B/2 = r/y$ and $\tan C/2 = r/z$. Also $a = y + z$, $b = z + x$ and $c = x + y$. It follows that

$$\begin{aligned}
a : b : c &= y + z : z + x : x + y \\
&= \left(\frac{1}{\tan B/2} + \frac{1}{\tan C/2}\right) : \left(\frac{1}{\tan C/2} + \frac{1}{\tan A/2}\right) : \left(\frac{1}{\tan A/2} + \frac{1}{\tan B/2}\right) \\
&= \left(\frac{v}{q} + \frac{w}{r}\right) : \left(\frac{w}{r} + \frac{u}{p}\right) : \left(\frac{u}{p} + \frac{v}{q}\right) \\
&= p(qw + vr) : q(pw + ru) : r(pv + qu) = fpu : gqv : hrw \ .
\end{aligned}$$

614. Determine those values of the parameter a for which there exist at least one line that is tangent to the graph of the curve $y = x^3 - ax$ at one point and normal to the graph at another.

Solution. The tangent at $(u, u^3 - au)$ has equation $y = (3u^2 - a)x - 2u^3$. This line intersects the curve again at the point whose abscissa is $-2u$ and whose tangent has slope $12u^2 - a$. The condition that the first tangent be normal at the second point is

$$(12u^2 - a)(3u^2 - a) = -1$$

or

$$36u^4 - 15u^2a + (a^2 + 1) = 0 \ .$$

The discriminant of this quadratic in u^2 is

$$225a^2 - 144(a^2 + 1) = 9(3a - 4)(3a + 4) \ .$$

The quadratic has positive real roots for u^2 if and only if $|a| \geq 4/3$.

615. The function $f(x)$ is defined for real nonzero x, takes nonzero real values and satisfies the functional equation

$$f(x) + f(y) = f(xyf(x + y)) \ ,$$

whenever $xy(x + y) \neq 0$. Determine all possibilities for f.

Solution. [J. Rickards] The functional equation is satisfied by $f(x) = 1/x$. More generally, suppose, if possible, that there exists a number a for which $f(a) = 1/b$ with $b \neq a$. Then

$$f(b) + f(a - b) = f(b(a - b)f(a)) = f(a - b) \ ,$$

whence $f(b) = 0$. But this contradicts the condition on f. Therefore there is no such a and $f(x) = 1/x$ is the unique solution.

616. Let T be a triangle in the plane whose vertices are lattice points (*i.e.*, both coordinates are integers), whose edges contain no lattice points in their interiors and whose interior contains exactly one lattice point. Must this lattice point in the interior be the centroid of the T?

Solution 1. [M. Valkov] Let ABC be the triangle and let X be the single lattice point within its interior. Using Pick's Theorem that the area of a lattice triangle is $(1/2)b + i - 1$, where b is the number of lattice points on the boundary and i the number in the interior, we find that $[ABC] = 3/2$ and $[ABX] = [BCX] = [CAX] = 1/2$. Let the line through X parallel to BC meet AB at Y and AC at Z. This is line is one-third of the distance from BC as A. Let AX meet BC at P. Then $YX : BP = AX : AP = 2 : 3$.

Since X is one-third the distance from AB as C, we have that $YX : BC = 1 : 3$, whence $2BP = BC$ and X is on the median from A. Similarly, X is on the other two medians and so is the centroid of the triangle.

Solution 2. [J. Schneider; J. Rickards] The answer is "yes". Without loss of generality, we can assume that the three points are $(0, 0)$, (a, b) and (u, v). The area of the triangle can be computed in two ways, by Pick's Theorem ($\frac{1}{2}b + i - 1$ where b is the number of lattice points on the boundary and i the number of lattice points in the interior of a polygon whose vertices are at lattice points) and directly using the formula for the area of a triangle with given vertices. This yields the equation

$$\frac{3}{2} = \frac{1}{2}|av - bu| \ ,$$

whence we deduce that $av - bu \equiv 0 \pmod 3$.

Since there is no lattice point in the interior of the sides of the triangle, it follows that, modulo 3, $a \equiv b \equiv 0$, $u \equiv v \equiv 0$ and $a \equiv u \& b \equiv v$ are each individually impossible. If $(a, b) \equiv (0, \pm 1)$, then $u \equiv 0$ and $v \equiv \mp 1$; thus, modulo 3, $a + u \equiv b + v \equiv 0$ and the centroid $(\frac{1}{3}(a + u), \frac{1}{3}(b + v))$ is a lattice point. Since the centroid lies inside the triangle and there is exactly one lattice point inside the triangle, the interior point must be the centroid. A similar analysis can be made if none of the coordinates a, b, u, v are divisible by 3. Thus, in all cases, the interior point is the centroid of the triangle.

617. Two circles are externally tangent at A and are internally tangent to a third circle Γ at points B and C. Suppose that D is the midpoint of the chord of Γ that passses through A and is tangent there to the two smaller given circles. Suppose, further, that the centres of the three circles are not collinear. Prove that A is the incentre of triangle BCD.

Solution 1. Let G denote the centre of the circle with points B and A on the circumference and H the centre of the circle with the points C and A on the circumference. Wolog, we assume that the former circle is the larger. Suppose that O is the centre of the circle Γ. The points G, A and H are collinear, as are B, G, O and C, H, O. Let the chord of Γ tangent to the smaller circles meet the circumference of Γ at J and K.

We have the OD and GH are both perpendicular to JK so that $GH \| OD$. Let BA and OD intersect at F. Since $BG = GA$ and triangles BGA and BOF are similar, $BO = OF$ and F lies on Γ. Similarly, the point E where CA and OD intersect lies on Γ. Since $\angle ABE = \angle FBE = 90° = \angle ADE$, the points B, E, D, A are concyclic. Therefore $\angle CBF = \angle CEF = \angle AED = \angle ABD$ and so A lies on the bisector of angle CBD. Similarly, A lies on the bisector of angle DCB. If follows that A is the incentre of triangle BCD.

Solution 2. Use the same notation as in the previous solution. Wolog, let the circle with centre G be at least as large as the circle with centre H. Suppose that the tangents to the circle Γ at B and C meet at the point L, and that LB and LA' are the tangents from L to the circle with centre G. Then $LC = LB = LA'$. There is a unique circle Δ that is tangent to LC and LA' at the points C and A'. This circle is tangent also to the circle Γ and the circle with centre G. Therefore, this circle must be the same circle with centre H, so that LA, LB and LC are each tangent to two of the three circles. Therefore, $LA = LB = LC$.

Observe that, because of subtended right angles, each of the quadrilaterals $LBOC$, $LDOB$, $LODC$ is concyclic. We have that

$$\angle LDC = \angle LOC = \angle LBC = \angle LCB = \angle LOB = \angle LDB \ ,$$

with the result that A lies on the bisector of angle BDC.

Let $\angle ABO = \beta$ and $\angle ACO = \gamma$. Then $\angle ACL = 90° - \gamma$, so that $\angle DLC = 2\gamma$. Similarly, $\angle DLB = 2\beta$. Therefore

$$\angle BLC = 2(\beta + \gamma) \Longrightarrow \angle BCL = 90° - \beta - \gamma \Longrightarrow \angle BCA = \angle ACL - \angle BCL = \beta \ .$$

Because $LODC$ is concyclic,

$$\angle OCD = \angle OLD = \angle OLC - \angle DLC = (\beta + \gamma) - 2\gamma = \beta - \gamma \ .$$

Hence

$$\angle ACD = \angle ACO + \angle OCD = \gamma + (\beta - \gamma) = \beta = \angle BCA$$

and A lies on the bisector of angle BCA. Therefore A is the incentre of triangle BCD.

618. Let a, b, c, m be positive integers for which $abcm = 1 + a^2 + b^2 + c^2$. Show that $m = 4$, and that there are actually possibilities with this value of m.

Solution. [J. Schneider] If any of a, b, c are even, then so is $abcm$. If a, b, c are all odd, then the right side of the equation is even and $abcm$ is even. Thus, $abcm$ must be even and an even number of a, b, c are even. If two of a, b, c are even, then the left side is congruent to 0 modulo 4 while the right is congruent to 2. Hence, it follows that all of a, b, c are odd. Therefore the right side is congruent to 4 modulo 8, and so m must be an odd multiple of 8.

If $m = 4$, then we have infinitely many solutions. One solution is $(m, a, b, c) = (4, 1, 1, 1)$. Suppose that we are given a solution $(m, a, b, c) = (4, 1, u, v)$. Then the equation is equivalent to $v^2 - 4uv + (2 - u^2) = 0$, *i.e.* v is a root of the quadratic equation

$$x^2 - 4ux + (2 - u^2) = 0 \ .$$

The second root $4u - v$ of this quadratic equation also yields a solution: $(m, a, b, c) = (4, 1, u, 4u - v)$. In this way, we can find an infinite sequence of solutions of the form $(m, a, b, c) = (4, 1, u_n, u_{n+1})$ where $u_1 = u_2 = 1$ and $u_{n+1} = 4u_n - u_{n-1}$.

Now suppose that $m \geq 12$. The equation can be rewritten

$$\frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab} + \frac{1}{abc} = m \ .$$

Wolog, let $a \leq b \leq c$. Then only the term c/ab is not less than 1, and we must have $c \geq 9ab$. Since

$$2 < (81a^2 - 1)(81b^2 - 1) = 81(81a^2b^2 - a^2 - b^2) + 1 \leq 81(c^2 - a^2 - b^2) + 1 \ ,$$

whence $c^2 > a^2 + b^2 + 1$.

Suppose that the given equation is solvable and that (m, a, b, c) is that solution which minimizes the sum $a + b + c$ for the given m. Since (m, a, b, x) satisfies the equation if and only if

$$x^2 - mbcx + (a^2 + b^2 + 1) = 0 \ ,$$

and since c is one root of this equation, the other root yields the solution $(m, a, b, (a^2 + b^2 + 1)/c)$. However, the last entry of this is less than c and yields a solution with a smaller sum. Thus, we have a contradiction. Therefore there are no solutions with $m > 4$.