Solutions for January

591. The point O is arbitrarily selected from the interior of the angle KAM. A line g is constructed through the point O, intersecting the ray AK at the point B and the ray AM at the point C. Prove that the value of the expression

$$\frac{1}{[AOB]} + \frac{1}{[AOC]}$$

does not depend on the choice of the line g. [*Note:* $[MNP]$ denotes the area of triangle MNP.]

Solution 1. Construct a line passing through the point O and parallel to AC. Let this line intersect the line AB at the point P. Taking note that two triangles having their bases on a line and their third vertex on a parallel line have areas in proportion to their bases, we obtain that

$$\begin{aligned}\frac{1}{[AOB]} + \frac{1}{[AOC]} &= \frac{[AOB]+[AOC]}{[AOB][AOC]} = \frac{[ABC]}{[AOB][AOC]} \\ &= \frac{[ABC]}{[AOB][AOC]} \cdot \frac{[APO]}{[APO]} = \frac{[ABC]}{[AOC]} \cdot \frac{[APO]}{[AOB]} \cdot \frac{1}{[APO]} = \frac{[ABC]}{[AOC]} \cdot \frac{|AP|}{|AB|} \cdot \frac{1}{[APO]} \\ &= \frac{[ABC]}{[AOC]} \cdot \frac{[APC]}{[ABC]} \cdot \frac{1}{[APO]} = \frac{[APC]}{[AOC]} \cdot \frac{1}{[APO]} = \frac{1}{[APO]} \ ,\end{aligned}$$

Since none of the points A, P, O depend on the position of the line g, the desired result follows.

Solution 2. Let $a = |AO|$, $b = |AB|$, $c = |AC|$, $\beta = \angle BAO$, $\gamma = \angle CAO$ and $\theta = \angle AOB$. The distance from O to AB is $a \sin\beta$ and from O to AC is $a\sin\gamma$. Therefore, $[AOB] = \frac{1}{2}ba\tan\beta$ and $[AOC] = \frac{1}{2}ca\tan\gamma$. Note that $\angle ABO = 180^\circ - (\theta + \beta)$ and $\angle ACO = \theta - \gamma$, so that, by the Law of Sines,

$$b = \frac{a\sin\theta}{\sin(\theta+\beta)} \qquad \text{and} \qquad c = \frac{a\sin\theta}{\sin(\theta-\gamma)} \ .$$

Therefore

$$\begin{aligned}\frac{1}{[AOB]} + \frac{1}{[AOC]} &= \frac{2}{ba\sin\beta} + \frac{2}{ca\sin\gamma} \\ &= \left(\frac{2}{a^2\sin\theta\sin\beta\sin\gamma}\right)(\sin(\theta+\beta)\sin\gamma + \sin(\theta-\gamma)\sin\beta) \\ &= \left(\frac{2}{a^2\sin\theta\sin\beta\sin\gamma}(\sin\theta\cos\beta\sin\gamma + \cos\theta sin\beta\sin\gamma + \sin\theta\cos\gamma\sin\beta - \cos\theta sin\gamma\sin\beta) \\ &= \left(\frac{2}{a^2\sin\beta\sin\gamma}(\cos\beta\sin\gamma + \cos\gamma\sin\beta) = 2a^{-2}(\cot\beta + \cot\gamma \ ,\end{aligned}$$

which does not depend on the variable quantities b, c and θ. The result follows.

592. The incircle of the triangle ABC is tangent to the sides BC, CA and AB at the respective points D, E and F. Points K from the line DF and L from the line EF are such that $AK\|BL\|DE$. Prove that:

(a) the points A, E, F and K are concyclic, and the points B, D, F and L are concyclic;

(b) the points C, K and L are collinear.

Solution. (a) Since AE is tanget to the circumcircle of triangle DEF and since $AK\|BL$,

$$\angle AEF = \angle EDF = \angle AKF \ ,$$

whence A, E, F, K are concyclic. Since BC is tangent to the circumcircle of triangle DEF and since $DE\|BL$,

$$\angle BDF = \angle FED = \angle LED = 180^\circ - \angle BLE = 180^\circ - \angle BLF \ ,$$

whence B, D, F, L are concyclic.

(b) Since $DE \| AK$, $AKEF$ is a concyclic quadrilateral and AB is tangent to circle DEF, we have that

$$\angle DEK = \angle EKA = \angle EFA = \angle EDK \ ,$$

whence $KD = KE$. Since $DE \| BL$, $BLFD$ is a concyclic quadrilateral and AB is tangent to circle DEF, we have that

$$\angle LDE = \angle BLD = \angle BFD = \angle LED \ ,$$

whence $LD = LE$. Since CD and CE are tangents to circle DEF, $CD = CE$. Therefore, all three points C, K, L lie on the right bisector of DE and so are collinear.

593. Consider all natural numbers M with the following properties:

(i) the four rightmost digits of M are 2008;

(ii) for some natural numbers $p > 1$ and $n > 1$, $M = p^n$.

Determine all numbers n for which such numbers M exist.

Solution. Since, modulo 10, squares are congruent to one of 0, 1, 4, 6, 9, and p^n is square for even values of n, there are no even values of n for which such a number M exists.

Since $p^n \equiv 2008 \pmod{10^4}$ implies that $p^n \equiv 8 \pmod{16}$, we see that p must be even. When p is divisible by 4, then $p^n \equiv 0 \pmod{16}$ for $n \geq 2$, and when p is twice an odd number, $p^n \equiv 0 \pmod{16}$ for $n \geq 4$. Therefore the only possibility for M is that it be the cube of a number congruent to 2 (mod 4).

The condition that $p^3 \equiv 2008 \pmod{10^4}$ implies that $p^3 \equiv 8 \pmod{125}$. Since

$$p^3 - 8 = (p-2)(p^2 + 2p + 4) = (p-2)[(p+1)^2 + 3] \ ,$$

and since the second factor is never divisible by 5 (the squares, modulo 5, are 0, 1, 4), we must have that $p \equiv 2 \pmod{125}$. Putting this together with p being twice an odd number, we find that the smallest possibilities are equal to 502 and 1002.

We have that $502^3 = 126506008$ and $1002^3 = 1006012008$. Thus, such numbers M exist if and only $n = 3$.

594. For each natural number N, denote by $S(N)$ the sum of the digits of N. Are there natural numbers N which satisfy the condition severally:

(a) $S(N) + S(N^2) = 2008$;

(b) $S(N) + S(N^2) = 2009$?

Solution. We have that

$$S(N) + S(N^2) \equiv N + N^2 = N(N+1)$$

(mod 9). This number is congruent to either 0 or 2, modulo 3. In particular, it can never assume the value of 2008, which is congruent to 1, modulo 3.

For part (b), we try a number N of the form

$$N = 1 + 10^3 + 10^6 + \cdots + 10^{3r} \ ,$$

where $100 \leq r \leq 999$. Then $S(N) = r + 1$,

$$N^2 = 1 + 2 \cdot 10^3 + 3 \cdot 10^6 + \cdots + r \cdot 10^{r-1} + (r+1) \cdot 10^r + r \cdot 10^{r+1} + \cdots + 2 \cdot 10^{6r-1} + 10^{6r}$$

and, since each coefficient of a power of 10 has at most three digits and there is no carry to a digit arising from another power,

$$S(n^2) = 2\sum_{k=1}^{r} S(k) + S(r+1) = 2\sum_{k=1}^{99} S(k) + 2\sum_{k=101}^{r} S(k) + S(r+1) \ .$$

The numbers less than 100 have 200 digits in all (counting 0 as the first digit of single-digit numbers), each appearing equally often (20 times), so that

$$2\sum_{k=1}^{99} S(k) = 2[20(1+2+\cdots+9)] = 1800 \ .$$

Now let $r = 108$. Then $S(100) + S(101) + S(108) = 9 + 36 = 45$, so that, when $N = 1001001\cdots 1001$ with 109 ones interspersed by double zeros,

$$S(N) + S(N^2) = 109 + 1800 + 90 + 10 = 2009 \ .$$

Therefore, the equation in (b) is solvable for some natural number N.

595. What are the dimensions of the greatest $n \times n$ square chessboard for which it is possible to arrange 111 coins on its cells so that the numbers of coins on any two adjacent cells (*i.e.* that share a side) differ by 1?

Solution. We begin by establishing some restrictions.The parity of the number of coins in any two adjacent cells differ, so that at least one of any pair of adjacent cells contains at least one coin. This ensures that the number of cells cannot exceed $2 \times 111 + 1 = 2 < 15^2$, so that $n \leq 14$. Since there are 111 cells, there must be an odd number of cells that contain an odd number of coins. Since in a 14×14 chessboard, there must be $98 = \frac{1}{2} \times 196$ cells with an odd number of coins, $n = 14$ is not possible.

We show that a 13×13 chessboard admits a suitable placement of coins. Begin by placing a single coin in every second cell so that each corner cell contains one coin. This uses up 85 coins. Now place two coins in each of thirteen of the remaining 84 vacant cells. We have placed $85 + 26 = 111$ coins in such a ways as to satisfy the condition.

Hence, a 13×13 chessboard is the largest that admits the desired placement.

596. A 12×12 square array is composed of unit squares. Three squares are removed from one of its major diagonals. Is it possible to cover completely the remaining part of the array by 47 rectangular tiles of size 1×3 without overlapping any of them?

Solution. Let the major diagonal in question go from upper left to lower right. Label the cells by letters A, B, C with A in the upper left corner, so that ABC appears in this cuyclic order across each row and ACB appears in this cyclic order down each column. There are thus 48 occurrences of each label, and each cell of the major diagonal is labelled with an A. Since each horizontal or vertical placement of 1×3 tiles must cover one cell with each label, any placement of any number of such tiles must cover equally many cells of each label. However, removing three cells down the major diagonal removes three cells of a single label and leaves of dearth of cells with label A. Therefore, a covering of the remaining 141 cells with 47 tiles is not possible.

597. Find all pairs of natural numbers (x, y) that satisfy the equation

$$2x(xy - 2y - 3) = (x + y)(3x + y) \ .$$

Solution. The given equation can be rewritten as a quadratic in y:

$$y^2 + (8x - 2x^2)y + (3x^2 + 6x) = 0 \ .$$

Its discriminant is equal to

$$(64x^2 - 32x^3 + 4x^4) - 4(3x^2 + 6x) = 4x(x^3 - 8x^2 + 13x - 6) = 4x(x-6)(x-1)^2 \ .$$

For there to be a solution in integers, it is necessary that this discriminant be a perfect square. This happens if and only of

$$z^2 = x(x-6) = (x-3)^2 - 9 \ ,$$

or

$$9 = (x-3)^2 - z^2 = (x+z-3)(x-z-3) \ ,$$

for some integer z. Checking all the factorizations $9 = (-9) \times (-1) = (-3) \times (-3) = (-1) \times (-9) = 9 \times 1 = 3 \times 3 = 1 \times 9$, we find that $(x, z) = (-2, \pm 4), (0, 0), (8, \pm 4), (6, 0)$.

This leads to a complete solutions set in integers:

$$(x, y) = (-2, 0), (-2, -8), (-, 0), (8, 4), (8, 60), (6, 12) \ .$$

Therefore, the only solutions in natural numbers to the equation are

$$(x, y) = (6, 12), (8, 4), (8, 60) \ ,$$

all of which check out.