Solutions

626. Let ABC be an isosceles triangle with $AB = AC$, and suppose that D is a point on the side BC with $BC > BD > DC$. Let BE and CF be diameters of the respective circumcircles of triangles ABD and ADC, and let P be the foot of the altitude from A to BC. Prove that $PD : AP = EF : BC$.

Solution 1. Since angles BDE and CDF are both right, E and F both lie on the perpendicular to BC through D. Since $ABDE$ and $ADCF$ are concyclic,

$$\angle AEF = \angle ABD = \angle ABC = \angle ACB = \angle ACD = \angle AFD = \angle AFE \ .$$

Therefore triangles AEF and ABC are similar. Thus AEF is isosceles and its altitude through A is perpendicular to DEF and parallel to BC, so that it is equal to PD. Therefore, from the similarity, $PD : AP = EF : BC$, as desired.

Solution 2. Since the chord AD subtends the same angle ($\angle ABC = \angle ACB$) in circles ABD and ACD, these circles must have equal diameters. The rotation with centre A that takes B to C takes the circle ABD to a circle with chord AC of equal diameter. The angle subtended at D by AB on the circumcircle of ABD is the supplement of the angle subtended at D by AC on the circumcircle of ACD. Therefore, this image circle must be the circle ACD. Therefore the diameter BE is carried to the diameter CF, and E is carried to F. Hence $AE = AF$ and $\angle BAC = \angle EAF$. Thus, triangles ABC and AEF are similar.

Now consider the composite of a rotation about A through a right angle followed by a dilatation of factor $|AE|/|AB|$. This transformation take B to E and C to F, and therefore the altitude AP to the altitude AM of triangle AEF which is therefore parallel to BC. Since D lies on the circumcircle of ABD with diameter BE, $\angle BDE = 90°$. Similarly, $\angle CDF = 90°$. Hence $AMDP$ is a rectangle and $AM = PD$. The result follows from the similarity of triangles ABC and AEF.

627. Let

$$f(x, y, z) = 2x^2 + 2y^2 - 2z^2 + \frac{7}{xy} + \frac{1}{z} \ .$$

There are three pairwise distinct numbers a, b, c for which

$$f(a, b, c) = f(b, c, a) = f(c, a, b) \ .$$

Determine $f(a, b, c)$. Determine three such numbers a, b, c.

Solution. Suppose that a, b, c are pairwise distinct and $f(a, b, c) = f(b, c, a) = f(c, a, b)$. Then

$$2a^2 + 2b^2 - 2c^2 + \frac{7}{ab} + \frac{1}{c} = 2b^2 + 2c^2 - 2a^2 + \frac{7}{bc} + \frac{1}{a}$$

so that

$$4(a^2 - c^2) = \left(\frac{1}{a} - \frac{1}{c}\right)\left(1 - \frac{7}{b}\right) = \frac{1}{abc}(c - a)(b - 7) \ .$$

Therefore $4abc(a + c) = 7 - b$. Similarly, $4abc(b + a) = 7 - c$. Subtracting these equations yields that $4abc(c - b) = c - b$ so that $4abc = 1$. It follows that $a + b + c = 7$.

Therefore

$$\begin{aligned} f(a, b, c) &= 2(a^2 + b^2) - 2c^2 + 28c + 4ab \\ &= 2(a + b)^2 - 2c^2 + 28c = 2(7 - c)^2 - 2c^2 + 28c \\ &= 98 - 28c + 2c^2 - 2c^2 + 28c = 98 \ . \end{aligned}$$

We can find such triples by picking any nonzero value of c and solving the quadratic equation $t^2 - (7 - c)t + (1/4c) = 0$ for a and b. For example, taking $c = 1$ yields the triple

$$(a, b, c) = \left(\frac{6 + \sqrt{35}}{2}, \frac{6 - \sqrt{35}}{2}, 1\right) \ .$$

628. Suppose that AP, BQ and CR are the altitudes of the acute triangle ABC, and that

$$9\overrightarrow{AP} + 4\overrightarrow{BQ} + 7\overrightarrow{CR} = \overrightarrow{O} \ .$$

Prove that one of the angles of triangle ABC is equal to $60°$.

Solution 1. [H. Spink] Since the sum of the three vectors $9\overrightarrow{AP}$, $4\overrightarrow{BQ}$, $7\overrightarrow{CR}$ is zero, there is a triangle whose sides have lengths $9|AP|$, $4|BQ|$, $7|CR|$ and are parallel to the corresponding vectors.

Where H is the orthocentre, we have that

$$\angle BHP = 90° - \angle QBC = \angle ACB$$

so that the angle between the vectors $\overrightarrow{AP}$ and $\overrightarrow{BQ}$ is equal to angle ACB. Similarly, the angle between vectors $\overrightarrow{BQ}$ and $\overrightarrow{CR}$ is equal to angle BAC. It follows that the triangle formed by the vectors is similar to triangle ABC and

$$|AB| : 7|CR| = |BC| : 9|AP| = |CA| : 4|BQ| \ .$$

Since twice the area of the triangle ABC is equal to

$$|AB| \times |CR| = |BC| \times |AP| = |CA| \times |BQ| \ ,$$

we have that (with conventional notation for side lengths)

$$\frac{c^2}{7} = \frac{a^2}{9} = \frac{b^2}{4}$$

so that $a : b : c = 3 : 2 : \sqrt{7}$.

If one angle of the triangle is equal to $60°$ we would expect it to be neither the largest nor the smallest. Accordingly, we compute the cosine of angle ACB, namely

$$\frac{a^2 + b^2 - c^2}{2ab} = \frac{9 + 4 - 7}{2 \times 3 \times 2} = \frac{6}{12} = \frac{1}{2} \ .$$

Therefore $\angle ACB = 60°$.

Solution 2. Let the angles of the triangle be $\alpha = \angle BAC$, $\beta = \angle CBA$ and $\gamma = \angle ACB$; let p, q, r be the respective magnitudes of vectors $\overrightarrow{AP}$, $\overrightarrow{BQ}$, $\overrightarrow{CR}$. Taking the dot product of the vector equation with $\overrightarrow{BC}$ and noting that $\angle QBC = 90 - \gamma$ and $\angle BCR = 90 - \beta$, we find that $4q \sin \gamma = 7r \sin \beta$. Similarly, $9p \sin \gamma = 7r \sin \alpha$ and $9p \sin \beta = 4q \sin \alpha$. Using the conventional notation for the sides of the triangle, we have that

$$a : b : c = \sin \alpha : \sin \beta : \sin \gamma = 9p : 4q : 7r \ .$$

However, we also have that twice the area of triangle ABC is equal to $ap = bq = cr$, so that $a : b : c = (1/p) : (1/q) : (1/r)$. Therefore $9p^2 = 4q^2 = 7r^2 = k$, for some constant k. Therefore

$$\cos \angle ACB = \frac{a^2 + b^2 - c^2}{2ab} = \frac{81p^2 + 16q^2 - 49r^2}{72pq}$$
$$= \frac{9k + 4k - 7k}{12k} = \frac{1}{2} \ ,$$

from which it follows that $\angle C = 60°$.

Solution 3. [C. Deng] Observe that

$$|BQ| = |BC| \cos \angle QBC = |BC| \angle \sin ACB \ ,$$

$$|CR| = |BC| \cos \angle RCB = |BC| \sin \angle ABC \ .$$

Resolving in the direction of $\overrightarrow{BC}$, we find from the given equation that

$$4|BC| \cos^2 \angle QBC = 4|BQ| \cos \angle QBC = 7|CR| \cos \angle RCB = 7|BC| \cos^2 \angle RCB$$

$$\Longrightarrow 4 \sin^2 \angle ACB = 7 \sin^2 \angle ABC \ .$$

By the Law of Sines, $AC : AB = \sin \angle ABC : \sin \angle ACB = 2 : \sqrt{7}$. Similarly $AC : BC = 2 : 3$, so that $CA : AB : BC = 2 : \sqrt{7} : 3$. The cosine of angle ACB is equal to $(4+9-7)/12 = 1/2$, so that $\angle ACB = 60°$.

629. (a) Let $a > b > c > d > 0$ and $a + d = b + c$. Show that $ad < bc$.

(b) Let a, b, p, q, r, s be positive integers for which

$$\frac{p}{q} < \frac{a}{b} < \frac{r}{s}$$

and $qr - ps = 1$. Prove that $b \geq q + s$.

(a) *Solution 1.* Since $c = a + d - b$, we have that

$$bc - ad = b(a + d - b) - ad = (a - b)b - (a - b)d = (a - b)(b - d) > 0 \ .$$

Solution 2. Let $a + d = b + c = u$. Then

$$bc - ad = b(u - b) - (u - d)d = u(b - d) - (b^2 - d^2) = (b - d)(u - b - d) \ .$$

Now $u = b + c > b + d$, so that $u - b - d > 0$ as well as $b - d > 0$. Hence $bc - ad > 0$ as desired.

Solution 3. Let $x = a - b > 0$. Since $a - b = c - d$, we have that $a = b + x$ and $d = c - x$. Hence

$$bc - ad = bc - (b + x)(c - x) = bx - cx + x^2 = x^2 + x(b - c) > 0 \ .$$

Solution 4. Since $\sqrt{a} > \sqrt{b} > \sqrt{c} > \sqrt{d}$, $\sqrt{a} - \sqrt{d} > \sqrt{b} - \sqrt{c}$. Squaring and using $a + d = b + c$ yields $2\sqrt{bc} > 2\sqrt{ad}$, whence the result.

(b) *Solution.* Since all variables represent integers,

$$aq - bp > 0, br - as > 0 \Longrightarrow aq - bp \geq 1, br - as \geq 1 \ .$$

Therefore

$$b = b(qr - ps) = q(br - as) + s(aq - bp) \geq q + s \ .$$

630. (a) Show that, if

$$\frac{\cos \alpha}{\cos \beta} + \frac{\sin \alpha}{\sin \beta} = -1 \ ,$$

then

$$\frac{\cos^3 \beta}{\cos \alpha} + \frac{\sin^3 \beta}{\sin \alpha} = 1 \ .$$

(b) Give an example of numbers α and β that satisfy the condition in (a) and check that both equations hold.

(a) *Solution 1.* Let

$$\lambda = \frac{\cos \beta}{\cos \alpha} \quad \text{and} \quad \mu = \frac{\sin \beta}{\sin \alpha} \ .$$

Since $\lambda^{-1} + \mu^{-1} = -1$, we have that $\lambda + \mu = -\lambda\mu$. Now

$$1 = \cos^2\beta + \sin^2\beta = \lambda^2\cos^2\alpha + \mu^2\sin^2\alpha = \lambda^2 + (\mu^2 - \lambda^2)\sin^2\alpha = \lambda^2 - (\mu - \lambda)\lambda\mu\sin^2\alpha \ .$$

Hence

$$\begin{aligned}
\frac{\cos^3\beta}{\cos\alpha} + \frac{\sin^3\beta}{\sin\alpha} &= \lambda^3\cos^2\alpha + \mu^3\sin^2\alpha \\
&= \lambda(\lambda^2\cos^2\alpha + \mu^2\sin^2\alpha) + (\mu - \lambda)\mu^2\sin^2\alpha \\
&= \lambda + (\mu - \lambda)\mu^2\sin^2\alpha \\
&= \frac{1}{\lambda}[\lambda^2 + (\lambda^2 - 1)\mu] \\
&= \frac{1}{\lambda}[\lambda^2 + \lambda^2\mu + \lambda + \lambda\mu \\
&= \lambda + \lambda\mu + 1 + \mu = 1 \quad .
\end{aligned}$$

Solution 2. [M. Boase]

$$\begin{aligned}
&\frac{\cos\alpha}{\cos\beta} + \frac{\sin\alpha}{\sin\beta} = -1 \Longrightarrow \\
&\sin(\alpha + \beta) + \sin\beta\cos\beta = 0 \ . \qquad (*)
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{\cos^3\beta}{\cos\alpha} + \frac{\sin^3\beta}{\sin\alpha} &= \frac{\cos\beta(1 - \sin^2\beta)}{\cos\alpha} + \frac{\sin\beta(1 - \cos^2\beta)}{\sin\alpha} \\
&= \frac{\cos\beta}{\cos\alpha} + \frac{\sin\beta}{\sin\alpha} - \sin\beta\cos\beta\left(\frac{\sin\beta}{\cos\alpha} + \frac{\cos\beta}{\sin\alpha}\right) \\
&= \frac{\sin(\alpha + \beta)}{\cos\alpha\sin\alpha} - \frac{\cos\beta\sin\beta(\cos(\alpha - \beta))}{\cos\alpha\sin\alpha} \\
&= \frac{-2\sin\beta\cos\beta + 2\sin(\alpha + \beta)\cos(\alpha - \beta)}{2\sin\alpha\cos\alpha} \qquad \text{using} \quad (*) \\
&= \frac{-2\sin\beta\cos\beta + [\sin 2\alpha + \sin 2\beta]}{\sin 2\alpha} = 1
\end{aligned}$$

since $2\sin\beta\cos\beta = \sin 2\beta$.

Solution 3. [A. Birka] Let $\cos\alpha = x$ and $\cos\beta = y$. Then

$$\frac{\sin\alpha}{\sin\beta} = \pm\sqrt{\frac{1 - x^2}{1 - y^2}} \quad .$$

Since

$$\frac{x}{y} + 1 = \mp\sqrt{\frac{1 - x^2}{1 - y^2}} \quad .$$

then

$$(x^2 + 2xy + y^2)(1 - y^2) = y^2(1 - x^2) \quad ,$$

whence

$$x^2 + 2xy = 2xy^3 + y^4 \quad .$$

Thus,

$$\begin{aligned}
\frac{\cos^3\beta}{\cos\alpha} + \frac{\sin^3\beta}{\sin\alpha} &= \frac{y^3}{x} \pm (1 - y^2)\sqrt{\frac{1 - y^2}{1 - x^2}} \\
&= \frac{y^3}{x} - \frac{(1 - y^2)y}{x + y} = \frac{y^4 + 2xy^3 - xy}{x(x + y)} \\
&= \frac{x^2 + xy}{x(x + y)} = 1 \quad .
\end{aligned}$$

Solution 4. [J. Chui] Note that the given equation implies that $\sin 2\beta = -2\sin(\alpha+\beta)$ and that the numerator of

$$\frac{\cos\alpha}{\cos\beta} + \frac{\sin\alpha}{\sin\beta} + \frac{\cos^3\beta}{\cos\alpha} + \frac{\sin^3\beta}{\sin\alpha}$$

is one quarter of

$$\begin{aligned}
4[\cos^2\alpha\sin\alpha\sin\beta &+ \sin^2\alpha\cos\alpha\cos\beta + \cos^4\beta\sin\alpha\sin\beta + \sin^4\beta\cos\alpha\cos\beta] \\
&= 4[\cos^2\alpha\sin\alpha\sin\beta + \sin^2\alpha\cos\alpha\cos\beta + (\cos^2\beta - \cos^2\beta\sin^2\beta)\sin\alpha\sin\beta \\
&\qquad + (\sin^2\beta - \sin^2\beta\cos^2\beta)\cos\alpha\cos\beta] \\
&= (4\cos^2\alpha + 4\cos^2\beta - \sin^2 2\beta)\sin\alpha\sin\beta + (4\sin^2\alpha + 4\sin^2\beta - \sin^2 2\beta)\cos\alpha\cos\beta \\
&= 2\sin 2\alpha\cos\alpha\sin\beta + 2\sin 2\beta\cos\beta\sin\alpha + 2\sin 2\alpha\sin\alpha\cos\beta + 2\sin 2\beta\cos\alpha\sin\beta \\
&\qquad - \sin^2 2\beta(\cos\alpha\cos\beta + \sin\alpha\sin\beta) \\
&= 2(\sin 2\alpha + \sin 2\beta)\sin(\alpha+\beta) - \sin^2 2\beta\cos(\alpha-\beta) \\
&= 2\sin(\alpha+\beta)[\sin 2\alpha + \sin 2\beta - 2\sin(\alpha+\beta)\cos(\alpha-\beta)] = 0 \quad ,
\end{aligned}$$

since

$$\sin 2\alpha + \sin 2\beta = \sin(\overline{\alpha+\beta} + \overline{\alpha-\beta}) + \sin(\overline{\alpha+\beta} - \overline{\alpha-\beta}) \ .$$

Solution 5. [A. Tang] From the given equation, we have that

$$\sin(\alpha+\beta) = -\sin\beta\cos\beta \quad ,$$

$$\frac{\cos\beta}{\cos\alpha} = \frac{-\sin\beta}{\sin\alpha + \sin\beta} \quad ,$$

and

$$\frac{\sin\beta}{\sin\alpha} = \frac{-\cos\beta}{\cos\alpha + \cos\beta} \quad .$$

Hence

$$\begin{aligned}
\frac{\cos^3\beta}{\cos\alpha} + \frac{\sin^3\beta}{\sin\alpha} &= \cos^2\beta\left[\frac{-\sin\beta}{\sin\alpha+\sin\beta}\right] + \sin^2\beta\left[\frac{-\cos\beta}{\cos\alpha+\cos\beta}\right] \\
&= -\frac{\sin\beta\cos\beta[\cos\alpha\cos\beta + \sin\alpha\sin\beta + 1]}{4\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)} \\
&= \frac{\sin(\alpha+\beta)[\cos(\alpha-\beta)+1]}{[2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha+\beta)][2\cos^2\frac{1}{2}(\alpha-\beta)]} = 1 \quad .
\end{aligned}$$

Solution 6. [D. Arthur] The given equations yield $2\sin(\alpha+\beta) = -\sin 2\beta$, $\cos\alpha\sin\beta = -\cos\beta(\sin\alpha + \sin\beta)$ and $\sin\alpha\cos\beta = -\sin\beta(\cos\alpha+\cos\beta)$. Hence

$$\begin{aligned}
\frac{\cos^3\beta}{\cos\alpha} + \frac{\sin^3\beta}{\sin\alpha} &= \frac{\cos^2\beta(\cos\beta\sin\alpha) + \sin^2\beta(\sin\beta\cos\alpha)}{\cos\alpha\sin\alpha} \\
&= \frac{-\cos^2\beta\sin\beta(\cos\alpha+\cos\beta) - \sin^2\beta\cos\beta(\sin\alpha+\sin\beta)}{\cos\alpha\sin\alpha} \\
&= \frac{-\cos\beta\sin\beta(\cos\alpha\cos\beta + \cos^2\beta + \sin\alpha\sin\beta + \sin^2\beta)}{\cos\alpha\sin\alpha} \\
&= \frac{-\sin 2\beta(1+\cos(\alpha-\beta))}{\sin 2\alpha} \\
&= \frac{-\sin 2\beta + 2\sin(\alpha+\beta)\cos(\alpha-\beta)}{\sin 2\alpha} \\
&= \frac{-\sin 2\beta + \sin 2\alpha + \sin 2\beta}{\sin 2\alpha} = 1 \quad .
\end{aligned}$$

Solution 7. [C. Deng] Let $\sin\beta = x$, $\cos\beta = y$, and $(\sin\alpha)/(\sin\beta) = c$. Thus, $(\cos\alpha)/(\cos\beta) = -1-c$. We have that

$$x^2 + y^2 = 1$$

and

$$(cx)^2 + (-1-c)y)^2 = 1 \ .$$

Solving the system yields that

$$x^2 = \frac{c^2+2c}{1+2c} \ , \quad y^2 = \frac{1-c^2}{1+2c} \ .$$

Therefore,

$$\begin{aligned}\frac{\sin^3\beta}{\sin\alpha} + \frac{\cos^3\beta}{\cos\alpha} &= \frac{x^2}{c} + \frac{y^2}{-1-c} = \frac{c^2+2c}{c(2c+1)} + \frac{1-c^2}{(-c-1)(2c+1)} \\ &= \frac{c+2}{2c+1} + \frac{c-1}{2c+1} = 1 \ .\end{aligned}$$

(b) Solution. The given equation is equivalent to $2\sin(\alpha+\beta) + \sin 2\beta = 0$. Try $\beta = -45^\circ$ so that $\sin(\alpha - 45^\circ) = \frac{1}{2}$. We take $\alpha = 75^\circ$. Now

$$\sin 75^\circ = \sin(45^\circ + 30^\circ) = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}+1}{2}\right)$$

and

$$\cos 75^\circ = \cos(45^\circ + 30^\circ) = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}-1}{2}\right) \ .$$

It is straightforward to check that both equations hold.

631. The sequence of functions $\{P_n\}$ satisfies the following relations:

$$P_1(x) = x \ , \qquad P_2(x) = x^3 \ ,$$

$$P_{n+1}(x) = \frac{P_n^3(x) - P_{n-1}(x)}{1 + P_n(x)P_{n-1}(x)} \ , \qquad n = 1, 2, 3, \cdots .$$

Prove that all functions P_n are polynomials.

Solution 1. Taking $x = 1, 2, 3, \cdots$ yields the respective sequences

$$\{1, 1, 0, -1, -1, 0, \cdots\} \ , \quad \{2, 8, 30, 112, 418, 1560, \cdots\} \ , \quad \{3, 27, 240, 2133, \cdots\} \ .$$

In each case, we find that

$$P_{n+1}(x) = x^2 P_n(x) - P_{n-1}(x) \tag{1}$$

for $n = 2, 3, \cdots$. If we can establish (1) in general, it will follow that all the functions P_n are polynomials.

From the definition of P_n, we find that

$$P_{n+1} + P_{n-1} = P_n(P_n^2 - P_{n+1}P_{n-1}) \ .$$

Therefore, it suffices to establish that $P_n^2 - P_{n+1}P_{n-1} = x^2$ for each n. Now, for $n \geq 2$,

$$\begin{aligned}[P_{n+1}^2 - P_{n+2}P_n] - [P_n^2 - P_{n+1}P_{n-1}] &= P_{n+1}(P_{n+1} + P_{n-1}) - P_n(P_{n+2} + P_n) \\ &= P_{n+1}P_n(P_n^2 - P_{n+1}P_{n-1}) - P_nP_{n+1}(P_{n+1}^2 - P_{n+2}P_n) \\ &= -P_{n+1}P_n[(P_{n+1}^2 - P_{n+2}P_n) - (P_n^2 - P_{n+1}P_{n-1})] \ ,\end{aligned}$$

so that either $P_{n+1}(x)P_n(x)+1 \equiv 0$ or $P_{n+1}^2 - P_{n+2}P_n = P_n^2 - P_{n+1}P_{n-1}$. The first identity is precluded by the case $x = 1$, where it is false. Hence

$$P_{n+1}^2 - P_{n+2}P_n = P_n^2 - P_{n+1}P_{n-1}$$

for $n = 2, 3, \cdots$. Since $P_2^2(x) - P_3(x)P_1(x) = x^2$, the result follows.

Solution 2. [By inspection, we make the conjecture that $P_n(x) = x^2 P_{n-1}(x) - P_{n-2}$. Rather than prove this directly from the rather awkward condition on P_n, we go through the back door.] Define the sequence $\{Q_n\}$ for $n = 0, 1, 2, \cdots$ by

$$Q_0(x) = 0 \ , \qquad Q_1(x) = x \ , \qquad Q_{n+1} = x^2 Q_n(x) - Q_{n-1}(x)$$

for $n \geq 1$. It is clear that $Q_n(x)$ is a polynomial of degree $2n-1$ for $n = 1, 2, \cdots$. We show that $P_n(x) = Q_n(x)$ for each n.

Lemma: $Q_n^2(x) - Q_{n+1}Q_{n-1} = x^2$ for $n \geq 1$.

Proof: This result holds for $n = 1$. Assume that it holds for $n = k - 1 \geq 1$. Then

$$\begin{aligned} Q_k^2(x) - Q_{k+1}(x)Q_{k-1}(x) &= Q_k^2(x) - (x^2 Q_k(x) - Q_{k-1}(x))Q_{k-1}(x) \\ &= Q_k(x)(Q_k(x) - x^2 Q_{k-1}(x)) + Q_{k-1}^2(x) \\ &= -Q_k(x)Q_{k-2}(x) + Q_{k-1}^2(x) = x^2 \ . \qquad \spadesuit \end{aligned}$$

From the lemma, we find that

$$\begin{aligned} &Q_{n+1}(x) + Q_{n-1}(x) + Q_{n+1}(x)Q_n(x)Q_{n-1}(x) \\ &\quad = x^2 Q_n(x) + Q_{n+1}(x)Q_n(x)Q_{n-1}(x) = Q_n(x)(x^2 + Q_{n+1}(x)Q_{n-1}(x)) = Q_n^3(x) \end{aligned}$$

$$\Longrightarrow Q_{n+1}(x) = \frac{Q_n^3(x) - Q_{n-1}(x)}{1 + Q_n(x)Q_{n-1}(x)} \qquad (n = 1, 2, \cdots) \ .$$

We know that $Q_1(x) = P_1(x)$ and $Q_2(x) = P_2(x)$. Suppose that $Q_n(x) = P_n(x)$ for $n = 1, 2, \cdots, k$. Then

$$Q_{k+1}(x) = \frac{Q_k^3(x) - Q_{k-1}(x)}{1 + Q_k(x)Q_{k-1}(x)} = \frac{P_k^3(x) - P_{k-1}(x)}{1 + P_k(x)P_{k-1}(x)} = P_{k+1}(x)$$

from the definition of P_{k+1}. The result follows.

Comment: It can also be established that $P_{n+1}^2 + P_n^2 = (1 + P_n P_{n+1})x^2$ for each $n \geq 0$.

Solution 3. [I. Panayotov] First note that the sequence $\{P_n(x)\}$ is defined for all values of x, *i.e.*, the denominator $1 + P_{n-1}(x)P_n(x)$ never vanishes for n and x. Suppose otherwise, and let n be the least number for which there exists u for which $1 + P_{n-1}(u)P_n(u) = 0$. Then $n \geq 3$ and

$$-1 = P_{n-1}(u)P_n(u) = \frac{P_{n-1}(u)^4 - P_{n-1}(u)P_{n-2}(u)}{1 + P_{n-1}(u)P_{n-2}(u)}$$

which implies that $P_{n-1}(u)^4 = -1$, a contradiction.

We now prove by induction that $P_{n+1} = x^2 P_n - P_{n-1}$. Suppose that $P_k = x^2 P_{k-1} - P_{k-2}$ for $3 \leq k \leq n$, so that in particular we know that P_k is a polynomial for $1 \leq k \leq n$. Substituting for P_k yields

$$P_{k-1}^3(x) = P_{k-1}(x)[x^2 + x^2 P_{k-1}(x)P_{k-2}(x) - P_{k-2}^2(x)]$$

for all x. If $P_{k-1}(x) \neq 0$, then

$$P_{k-1}^2(x) = x^2 + x^2 P_{k-1}(x)P_{k-2}(x) - P_{k-2}^2(x) \ .$$

Both sides of this equation are polynomials and so continuous functions of x. Since the roots of P_{k-1} constitute a finite discreet set, this equation holds when x is one of the roots as well. Now

$$\begin{aligned} P_{n+1} &= \frac{P_n^3 - P_{n-1}}{1 + P_n P_{n-1}} = \frac{P_n(x^2 P_{n-1} - P_{n-2})^2 - P_{n-1}}{1 + P_n P_{n-1}} \\ &= \frac{P_n(x^4 P_{n-1}^2 - x^2 P_{n-1} P_{n-2} + x^2 - P_{n-1}^2) - P_{n-1}}{1 + P_n P_{n-1}} \\ &= \frac{P_n(x^2 P_n P_{n-1} + x^2 - P_{n-1}^2) - P_{n-1}}{1 + P_n P_{n-1}} \qquad \text{since} \quad x^2 P_{n-1} - P_{n-2} = P_n \\ &= \frac{(x^2 P_n - P_{n-1})(1 + P_n P_{n-1})}{1 + P_n P_{n-1}} = x^2 P_n - P_{n-1} \quad . \end{aligned}$$

The result follows.

632. Let a, b, c, x, y, z be positive real numbers for which $a \leq b \leq c$, $x \leq y \leq z$, $a + b + c = x + y + z$, $abc = xyz$, and $c \leq z$, Prove that $a \leq x$.

Solution. Let

$$p(t) = (t - a)(t - b)(t - c) = t^3 - (a + b + c)t^2 + (ab + bc + ca)t - abc$$

and

$$q(t) = (t - x)(t - y)(t - z) = t^3 - (x + y + z)t^2 + (xy + yz + zx)t - xyz \ .$$

Then $p(t) - q(t) = (ab + bc + ca - xy - yz - zx)t$ never changes sign for positive values of t. Since $p(t) > 0$ for $t > c$, we have that $p(z) - q(z) = p(z) \geq 0$, so that $p(t) \geq q(t)$ for all $t > 0$.

Hence, for $0 < t < a$, we have that $q(t) \leq p(t) < 0$, from which it follows that $q(t)$ has no root less than a. Hence $x \geq a$ as desired.