

Solutions for October

640. Suppose that $n \geq 2$ and that, for $1 \leq i \leq n$, we have that $x_i \geq -2$ and all the x_i are nonzero with the same sign. Prove that

$$(1 + x_1)(1 + x_2) \cdots (1 + x_n) > 1 + x_1 + x_2 + \cdots + x_n \quad ,$$

Solution 1. When $n = 2$, we have that

$$(1 + x_1)(1 + x_2) = 1 + x_1 + x_2 + x_1x_2 > 1 + x_1 + x_2$$

since x_1 and x_2 are nonzero with the same sign. Suppose, as an induction hypothesis, that the result holds for $n = k \geq 2$. Then

$$\begin{aligned} & (1 + x_1)(1 + x_2) \cdots (1 + x_k)(1 + x_{k+1}) - (1 + x_1 + x_2 + \cdots + x_k + x_{k+1}) \\ &= [(1 + x_1)(1 + x_2) \cdots (1 + x_k) - (1 + x_1 + x_2 + \cdots + x_k)] \\ &\quad + x_{k+1}[(1 + x_1)(1 + x_2) \cdots (1 + x_k) - 1] \\ &> x_{k+1}[(1 + x_1)(1 + x_2) \cdots (1 + x_k) - 1] \equiv A . \end{aligned}$$

If $x_i > 0$ ($1 \leq i \leq k + 1$), then $1 + x_i > 1$ ($1 \leq i \leq k$) and $A > 0$.

Let $-2 \leq x_i < 0$. Then, for $1 \leq i \leq k$,

$$\begin{aligned} -1 \leq 1 + x_i < 1 &\implies -1 \leq (1 + x_1)(1 + x_2) \cdots (1 + x_k) \leq 1 \\ &\implies (1 + x_1)(1 + x_2) \cdots (1 + x_k) - 1 \leq 0 . \end{aligned}$$

Since also $x_{k+1} < 0$, $A \geq 0$. Hence

$$(1 + x_1)(1 + x_2) \cdots (1 + x_{k+1}) > 1 + x_1 + x_2 + \cdots + x_{k+1} .$$

The result follows by induction.

Solution 2. The case $n = 2$ is proved as in the first solution. Suppose that all x_i are negative and at least two, say x_1 and x_2 lie in $[-2, -1]$. Then

$$(1 + x_1)(1 + x_2) \cdots (1 + x_n) \geq -1 \geq 1 - 1 - 1 \geq 1 + x_1 + x_2 + \cdots + x_n$$

since $-2 \leq x_i < 0$ and $|1 + x_i| \leq 1$ for $1 \leq i \leq n$.

Henceforth assume that either (i) all x_i are positive ($1 \leq i \leq n$) or (ii) all x_i are negative with $-2 \leq x_1 < 0$ and $-1 < x_i < 0$ for $2 \leq i \leq n$. As an induction hypothesis, assume that the result holds for $n = k \geq 2$. Then $1 + x_{k+1} > 0$, so that

$$\begin{aligned} & (1 + x_1)(1 + x_2) \cdots (1 + x_k)(1 + x_{k+1}) \\ &> (1 + x_1 + x_2 + \cdots + x_k)(1 + x_{k+1}) \quad \text{by the induction hypothesis} \\ &> 1 + x_1 + x_2 + \cdots + x_k + x_{k+1} \quad \text{by the } n = 2 \text{ case.} \end{aligned}$$

The result follows by induction.

641. Observe that $x^2 + 5x + 6 = (x + 2)(x + 3)$ while $x^2 + 5x - 6 = (x + 6)(x - 1)$. Determine infinitely many coprime pairs (m, n) of positive integers for which both $x^2 + mx + n$ and $x^2 + mx - n$ can be factored as a product of linear polynomials with integer coefficients.

Solution 1. For the factorizations to occur, both discriminants must be squares: $m^2 - 4n = u^2$, $m^2 + 4n = v^2$ for some integers u and v . Suppose m^2 can be expressed as the sum of two squares: $m^2 = p^2 + q^2$. Then $2m^2 = (p + q)^2 + (p - q)^2$. Write $u = p - q$, $v = p + q$. Then $8n = v^2 - u^2 = 4pq$ so that $n = \frac{1}{2}pq$.

Now we construct our examples. Let r and s be a coprime pair of integers with opposite parity. Define $p = r^2 - s^2$, $q = 2rs$, $m = r^2 + s^2$ and $n = rs(r^2 - s^2)$. Then any prime power divisor of m and n must divide both $r^2 + s^2$ and one of r , s and $r^2 - s^2$, and hence both $2r^2$ and $2s^2$. Hence it must divide 2. But we have arranged for m to be odd. Hence m and n are coprime. Observe that

$$x^2 + (r^2 + s^2)x + rs(r^2 - s^2) = (x + s(r + s))(x + r(r - s))$$

and

$$x^2 + (r^2 + s^2)x - rs(r^2 - s^2) = (x + r(r + s))(x - s(r - s)) .$$

Solution 2. With the notation of the first solution, we have that $m^2 - u^2 = v^2 - m^2$, whence $v^2 - 2m^2 = -u^2$. Let us take $u = 1$. We show that $v^2 - 2m^2 = -1$ has infinitely many solutions with m odd. Let $(v, m) = (v_k, m_k)$ where

$$(v_1, m_1) = (1, 1) \quad \text{and} \quad v_{k+1} + m_{k+1}\sqrt{2} = (3 + 2\sqrt{2})(v_k + m_k\sqrt{2})$$

so

$$v_{k+1} = 3v_k + 4m_k \quad m_{k+1} = 2v_k + 3m_k$$

for $k \geq 1$. By induction, it is proved that, for each k , $v_k^2 - 2m_k^2 = -1$. Let $(m, n) = (m_k, \frac{1}{4}(m_k^2 - 1))$. Then it is readily shown that (m, n) are both integers and satisfy the condition of the problem. For example, we have

$$(u, v; m, n) = (1, 1; 1, 0), (1, 7; 5, 6), (1, 41; 29, 210), \dots$$

so that, for example, $x^2 + 29x + 210 = (x + 14)(x + 15)$ and $x^2 + 29x - 210 = (x + 35)(x - 6)$.

Solution 3. [J. Rickards, M. Boase] Let a be even and let $(m, n) = (a^2 + 1, a^3 - a)$. Then the greatest common divisor of m and a is 1, as is the greatest common divisor of m and $a^2 - 1$. Then

$$x^2 + (a^2 + 1)x + (a^3 - a) = (x + \overline{a^2 - a})(x + \overline{a + 1})$$

and

$$x^2 + (a^2 + 1)x - (a^3 - a) = (x + \overline{a^2 + a})(x - \overline{a - 1}) .$$

Comment. This is a special case of Solution 1.

Solution 4. [S. Hemmati] Let k be an integer and let

$$m = 4k^2 + 1 = (2k + 1)^2 - 4k = (2k - 1)^2 + 4k$$

and $n = 2k(2k + 1)(2k - 1)$. The three factors of n are pairwise coprime, and it follows that the greatest common divisor of m and n is 1. We have that

$$x^2 + mx - n = [x + 2k(2k + 1)][x - (2k - 1)]$$

and

$$x^2 + mx + n = [x + 2k(2k - 1)][x + (2k + 1)] .$$

Solution 5. [C. Deng] Suppose that $x^2 + mx - n$ has roots a and $-b$ and that $x^2 + mx + n$ has roots r and s . Then $n = ab = rs$ and $m = b - a = -r - s$. In terms of a and b , the values of r and s are given by

$$\frac{a - b \pm \sqrt{a^2 - 6ab + b^2}}{2} .$$

Since $a - b$ and $a^2 - 6ab + b^2$ have the same parity, these are integers if and only if $a^2 - 6ab + b^2$ is a square. Any values of a and b which make this quantity square will yield acceptable values of m and n .

Let $(a_1, b_1) = (1, 6)$, and for $k \geq 1$,

$$a_{k+1} = 6a_k - b_k, \quad b_{k+1} = a_k.$$

Then

$$a_{k+1}^2 - 6a_{k+1}b_{k+1} + b_{k+1}^2 = (6a_k - b_k)^2 - 6(6a_k - b_k)a_k + a_k^2 = a_k^2 - 6a_kb_k + b_k^2,$$

so that, by induction, we see that this quantity is equal to 1 for all $k \geq 1$. Thus

$$(r_k, s_k) = \left(\frac{a_k - b_k - 1}{2}, \frac{a_k - b_k + 1}{2} \right).$$

Observe that the greatest common divisor of a_{k+1} and b_{k+1} is equal to that of a_k and b_k , and so, by induction, equal to 1, the greatest common divisor of 1 and 6. It follows, for all k , that $a_k b_k$ and $a_k - b_k$ are relatively prime. Thus, the pair

$$(x^2 + (b_k - a_k)x - a_k b_k, x^2 + (b_k - a_k)x + a_k b_k)$$

satisfy the desired conditions for $k \geq 1$.

In particular, we find that

$$\begin{aligned} x^2 + 5x - 6 &= (x + 6)(x - 1), & x^2 + 5x + 6 &= (x + 2)(x + 3); \\ x^2 + 29x - 210 &= (x + 35)(x - 6), & x^2 + 29x + 210 &= (x + 14)(x + 15); \\ x^2 + 169x - 7140 &= (x + 204)(x - 35), & x^2 + 169x + 7140 &= (x + 84)(x + 85). \end{aligned}$$

Solution 5. [K. Zhou] Suppose that $x^2 + mx - n$ has integer roots $-a$ and $-b$ and that $x^2 + mx + n$ has an integer root c . Then

$$x^2 + mx - n = (x + a)(x + b)$$

so that $m = a + b$ and $n = -ab$. The two roots of the polynomial $x^2 + mx + n$ are $-c$ and $c - m$, where $-ab = n = c(a + b - c)$. Therefore $a(b + c) = c^2 - bc$ so that

$$a = c - 2b + \frac{2b^2}{b + c}.$$

Thus, $b + c$ must divide $2b^2$ as all the other terms in the equation are integers.

To construct our examples, let a, b, c be chosen so that a and b are integers, $b + c = 1$ and $a = c - 2b + 2b^2$. Then c is an integer and

$$c = 1 - b, \quad a = 1 - 3b + 2b^2 = (1 - b)(1 - 2b).$$

Therefore, let

$$\begin{aligned} m &= a + b = 1 - 2b + 2b^2 = (1 - b)^2 + b^2 \\ &= (1 - b) + b(2b - 1) \end{aligned}$$

and

$$n = -ab = -b(1 - b)(1 - 2b).$$

Suppose, if possible, that some prime p divides both m and n . From the factorization of n , it follows that p divides one of b , $1 - b$ and $1 - 2b$, and from the expressions for m , we see that it must divide all three of

these numbers. But this is impossible, as b and $1 - b$ are coprime. Therefore, for all integers b , m and n are coprime.

We verify that these values of m and

$$x^2 + (1 - 2b + b^2)x + b(b - 1)(2b - 1) = (x + b)(x + (b - 1)(2b - 1)) ;$$

$$x^2 + (1 - 2b + b^2)x - b(b - 1)(2b - 1) = (x - b + 1)(x + b(2b - 1)) .$$

- 642.** In a convex polyhedron, each vertex is the endpoint of exactly three edges and each face is a concyclic polygon. Prove that the polyhedron can be inscribed in a sphere.

Solution. Let us begin with a couple of preliminary observations. Since three edges are incident with each vertex, exactly three faces of the polyhedron meet at each vertex. The centre of the circumscribing circle of any face is the point common to the right bisectors of the edges. The planes that right bisect the edges of a face intersect in a line perpendicular to the face, and this line is the locus of the centres of spheres which contain all the vertices of the face. Finally, any two vertices of the polyhedron can be joined by a path of edges of the polyhedron.

Any two adjacent faces of the polyhedron are inscribed in a unique sphere. Let the edge AB be common to two faces α and β which have respective circumcentres P and Q . The respective lines m and n to these faces through their circumcentres are non-parallel lines on the plane right-bisecting AB and so intersect in a unique point. This point is the centre of the only sphere that contains all of the vertices of α and β .

The three faces meeting at any vertex are contained in the same sphere. Suppose that vertex A belongs to the edges AB , AC and AD . The right bisecting planes of the edges AB and AC meet in a line through the centre of and perpendicular to the circumcircle of ABC . The right bisecting plane of AD is not parallel to this line and does not contain it, and so meets it in a single point. This point, lying on the perpendicular to each of the three faces adjacent to A and passing through their circumcentres is equidistance from all the vertices of these faces and so is the centre of a sphere containing these faces.

There is a circumscribing sphere for the polyhedron. Suppose this is false. Then there must be two vertices for which the spheres circumscribing the faces about the vertices differ. Join the two vertices by a path of edges. For one of the edges, say RS , the sphere circumscribing the faces meeting at R must be different from the sphere circumscribing the faces meeting at S . But then this means that the two faces adjacent to RS must be circumscribed by two separate spheres, contrary to what was shown above. Hence the desired result follows.

- 643.** Let n^2 distinct integers be arranged in an $n \times n$ square array ($n \geq 2$). Show that it is possible to select n numbers, one from each row and column, such that if the number selected from any row is greater than another number in this row, then this latter number is less than the number selected from its column.

Solution. We proceed in a number of rounds. In Round 1, select the least element in each row. If each column has one such number, we stop; otherwise, deselect in any column all but the largest of the selected numbers. Any row that does not contain a selected number, we call *free*. In each subsequent round, pick the least element not yet tried from each free row, and then deselect all but the biggest number in each column. Since any row can be freed at most $n - 1$ times, there are at most $n(n - 1) + 1$ rounds. In the final round, each column must have exactly one element.

Example:

$$\begin{array}{ccc} 6 & \mathbf{5} & 11 \\ 4 & \mathbf{2} & 3 \\ 7 & 10 & \mathbf{1} \end{array} \quad \rightarrow \quad \begin{array}{ccc} 6 & \mathbf{5} & 11 \\ 4 & 2 & \mathbf{3} \\ 7 & 10 & \mathbf{1} \end{array} \quad \rightarrow \quad \begin{array}{ccc} 6 & \mathbf{5} & 11 \\ 4 & 2 & \mathbf{3} \\ \mathbf{7} & 10 & 1 \end{array}$$

Suppose, wolog (by shuffling the columns if necessary), that the entries $a_{11}, a_{22}, \dots, a_{nn}$ are selected from the array (a_{ij}) . If $a_{ik} < a_{ii}$, then this number must have been an earlier possible selection and was rejected in favour of a larger number in its column. Hence $a_{ik} < a_{kk}$.

Comment: There is a dual procedure taking the largest element of each column and rejecting all but the smallest selected number in each row.

- 644.** Given a point P , a line \mathcal{L} and a circle \mathcal{C} , construct with straightedge and compasses an equilateral triangle PQR with one vertex at P , another vertex Q on \mathcal{L} and the third vertex R on \mathcal{C} .

Solution 1. Analysis. Suppose that we have the required triangle PQR with $Q \in \mathcal{L}$ and $R \in \mathcal{C}$. Then a 60° rotation with centre P takes \mathcal{C} to a circle \mathcal{C}' and R to the point lying on the intersection of \mathcal{C}' and \mathcal{L} . Accordingly, we need to construct a rotated image \mathcal{C}' of \mathcal{C} and, if this intersects \mathcal{L} , then we can construct the triangle.

Construction. Let O be the centre of \mathcal{C} . Construct an equilateral triangle POO' and with centre O' construct a circle \mathcal{C}' with radius equal to that of \mathcal{C} . If this circle \mathcal{C}' intersects \mathcal{L} at R , then there are two constructible points which with P and R are the vertices of an equilateral triangle; one of them Q will lie on \mathcal{C} .

Proof. The circle \mathcal{C}' is the image of \mathcal{C} under a 60° rotation with centre P that carries O to O' . The point R lies on \mathcal{C}' so its inverse image Q under the rotation lies on \mathcal{C} . Since $PQ = PR$ and $\angle P = 60^\circ$, PQR is an equilateral triangle.

Comment. There are two possible images of \mathcal{C} yielding up to four possibilities for R . However, it is also possible that neither images intersects \mathcal{L} and the construction is not possible.

- 645.** Let $n \geq 3$ be a positive integer. Are there n positive integers a_1, a_2, \dots, a_n not all the same such that for each i with $3 \leq i \leq n$ we have

$$a_i + S_i = (a_i, S_i) + [a_i, S_i] .$$

where $S_i = a_1 + a_2 + \dots + a_i$, and where (\cdot, \cdot) and $[\cdot, \cdot]$ represent the greatest common divisor and least common multiple respectively?

Solution 1. Letting $b_i = (a_i, S_i)$, we find that

$$[a_i, S_i] = \frac{a_i S_i}{(a_i, S_i)} = \frac{a_i S_i}{b_i} .$$

The given condition is equivalent to $a_i + S_i = b_i + (a_i S_i / b_i)$, which is equivalent to

$$0 = b_i^2 - (a_i + S_i)b_i + a_i S_i = (b_i - a_i)(b_i - S_i) .$$

We can achieve the condition by making $a_i = (a_i, S_i)$ and $S_i = [a_i, S_i]$. Let $a_1 = a_2 = 1$, $a_i = 2^{i-2}$ for $i \geq 3$. Then

$$\begin{aligned} S_i &= 1 + \sum_{j=2}^i 2^{j-2} = 1 + (2^{i-1} - 1) = 2^{i-1} \\ \implies (a_i, S_i) &= 2^{i-2} , \quad [a_i, S_i] = 2^{i-1} \end{aligned}$$

for $i \geq 3$.

Solution 2. Let $a_i = 1$, $a_2 = 2$ and $a_i = 3 \cdot 2^{i-3}$ for $i \geq 3$. Then

$$S_i = 1 + 2 + 3 \sum_{j=0}^{i-3} 2^j = 1 + 2 + 3(2^{i-2} - 1) = 3 \cdot 2^{i-2}$$

$$\implies (a_i, S_i) = 3 \cdot 2^{i-3}, \quad [a_i, S_i] = 3 \cdot 2^{i-2}$$

for $i \geq 3$.

Solution 3. [K. Purbhoo] Choose a_1 at will, and let $a_i = S_{i-1}$ for $i \geq 2$. Then $S_i = S_{i-1} + a_i = 2a_i$, $a_i + S_i = 3a_i$, $(a_i, S_i) = a_i$ and $[a_i, S_i] = 2a_i$ for $i \geq 2$.

Solution 4. [K. Yeats] Let $a_1 = 1$, $a_2 = 3$ and $a_n = 2^{n-1}$ for $n \geq 3$. Then $S_n = 2^n$ for $n \geq 2$ and, for each i with $3 \leq i \leq n$, $(a_i, S_i) = 2^{i-1}$ and $[a_i, S_i] = 2^i$.

- 646.** Let ABC be a triangle with incentre I . Let AI meet BC at L , and let X be the contact point of the incircle with the line BC . If D is the reflection of L on X , we construct B' and C' as the reflections of D with respect to the lines BI and CI , respectively. Show that the quadrilateral $BCC'B'$ is cyclic.

Solution 1. Without loss of generality, we may assume that $AC \geq AB$. Observe that B' lies on the side AB and C' lies on the side AC . We use the conventional notation for the sides a, b, c of the triangle and $2s = a + b + c$ for the perimeter. Let Z and Y be the tangent points of the incircle with sides AB and AC respectively. Observe that $IX \perp BC$, $IY \perp AC$ and $IZ \perp AB$.

We have that $XL = XD = ZB' = YC'$,

$$\begin{aligned} |XC| &= s - c, & |AY| &= |AZ| = s - a; \\ |LC| &= \frac{ab}{b+c}. \end{aligned}$$

Therefore

$$\begin{aligned} |XL| &= s - c - \frac{ab}{b+c}, \\ |AB'| &= |AZ| + |ZB| \\ &= s - a + s - c - \frac{ab}{b+c} = b - \frac{ab}{b+c} \\ &= b \left(1 - \frac{a}{b+c} \right); \end{aligned}$$

and

$$\begin{aligned} |AC'| &= |AY| - |YC'| \\ &= s - a - s + c + \frac{ab}{b+c} \\ &= \frac{-ab + bc - ac + c^2 + ab}{b+c} \\ &= c \left(\frac{b+c-a}{b+c} \right) = c \left(1 - \frac{a}{b+c} \right). \end{aligned}$$

Therefore

$$AC' : AB' = c : b = AB : AC$$

so that triangles ABC and $AC'B'$ are similar and $\angle ABC = \angle AC'B'$, $\angle ACB = \angle AB'C'$. Therefore the quadrilateral $BCC'B'$ is concyclic.

Solution 2. [S. Sun] Suppose that $AC \geq AB$. Use the same notation as in Solution 1, and let $t = |BL|$. Then $|ZB'| = |YC'| = |DX| = |XL| = t - (s - b)$. We have that

$$|AB'| = (s - a) + t - (s - b) = t + (b - a)$$

and

$$|AC'| = (s - a) - t + (s - b) = c - t,$$

whereupon

$$AB' : AC' = [t + (b - a)] : [c - t] = [b - (a - t)] : [c - t] .$$

However, as AL is an angle bisector, we have that $(a - t) : t = b : c$, so that

$$[b - (a - t)] : [c - t] = b : c = AC : AB .$$

Therefore, triangles $AC'B'$ and ABC are similar, and we can conclude, as in Solution 1, that $BCC'B'$ is concyclic.

Comment. Notice that Solutions 1 and 2 follow the same strategy, but the second solution is cleaner as it avoids the actual computation of t and merely exploits a relationship involving this variable.

Solution 3. [A. Murali] We again assume that $AC \geq AB$ and use the notation of Solution 1. We first show that $AB'IC'$ is concyclic. Observe that $\angle ZB'I = \angle LD'I = \angle YC'I$, so that triangles $IB'Z$ and $IC'Y$ are similar and $\angle ZIB' = \angle YIC'$. Thus $\angle B'AC' = 180^\circ - \angle ZIY = 180^\circ - \angle B'IC'$ and $AB'IC'$ is concyclic. It follows that $\angle B'C'I = \angle BAI = \frac{1}{2}\angle BAC = \angle LAC$.

We have that

$$\angle CC'I = \angle IDC = \angle ILB = \angle LAC + \angle ACL ,$$

whence

$$\angle CC'B' = \angle CC'I + \angle B'C'I = (\angle LAC + \angle ACL) + \angle LAC = \angle ACB + \angle BAC = 180^\circ - \angle ABC .$$

Therefore, $BCC'B'$ is concyclic.