PROBLEMS FOR MARCH

Please send your solutions to

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individually as you solve the problems. Electronic files can be sent to *barbeau@math.utoronto.ca*. However, please do not send scanned files; they use a lot of computer space, are often indistinct and can be difficult to download.

It is important that your complete mailing address and your email address appear legibly on the front page. If you do not write your family name last, please underline it.

668. The nonisosceles right triangle ABC has $\angle CAB = 90^\circ$. The inscribed circle with centre T touches the sides AB and AC at U and V respectively. The tangent through A of the circumscribed circle meets UV produced in S. Prove that

(a) $ST \parallel BC$;

(b) $|d_1 - d_2| = r$, where r is the radius of the inscribed circle and d_1 and d_2 are the respective distances from S to AC and AB.

669. Let $n \geq 3$ be a natural number. Prove that

$$1989 | n^{n^{n^n}} - n^{n^n} \quad ,$$

i.e., the number on the right is a multiple of 1989.

670. Consider the sequence of positive integers $\{1, 12, 123, 1234, 12345, \cdots\}$ where the next term is constructed by lengthening the previous term at the right-hand end by appending the next positive integer. Note that this next integer occupies only one place, with "carrying"occurring as in addition. Thus, the ninth and tenth terms of the sequence are 123456789 and 1234567900 respectively. Determine which terms of the sequence are divisible by 7.

671. Each point in the plane is coloured with one of three distinct colours. Prove that there are two points that are unit distant apart with the same colour.

672. The Fibonacci sequence $\{F_n\}$ is defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n = 0, \pm 1, \pm 2, \pm 3, \cdots$. The real number τ is the positive solution of the quadratic equation $x^2 = x + 1$.

(a) Prove that, for each positive integer n, $F_{-n} = (-1)^{n+1} F_n$.

(b) Prove that, for each integer n, $\tau^n = F_n \tau + F_{n-1}$.

(c) Let G_n be any one of the functions $F_{n+1}F_n$, $F_{n+1}F_{n-1}$ and F_n^2. In each case, prove that $G_{n+3} + G_n = 2(G_{n+2} + G_{n+1})$.

673. ABC is an isosceles triangle with $AB = AC$. Let D be the point on the side AC for which $CD = 2AD$. Let P be the point on the segment BD such that $\angle APC = 90^\circ$. Prove that $\angle ABP = \angle PCB$.

674. The sides BC, CA, AB of triangle ABC are produced to the poins R, P, Q respectively, so that $CR = AP = BQ$. Prove that triangle PQR is equilateral if and only if triangle ABC is equilateral.