Solutions For May

675. ABC is a triangle with circumcentre O such that $\angle A$ exceeds $90°$ and $AB < AC$. Let M and N be the midpoints of BC and AO, and let D be the intersection of MN and AC. Suppose that $AD = \frac{1}{2}(AB + AC)$. Determine $\angle A$.

Solution. Assign coordinates: $A \sim (0,0)$, $B \sim (2\cos\theta, 2\sin\theta)$, $C \sim (2u, 0)$ where $90° < \theta < 180°$ and $u > 1$. First, we determine O as the intersection of the right bisectors of AB and AC. The centre of AB has coordinates $(\cos\theta, \sin\theta)$ and its right bisector has equation

$$(\cos\theta)x + (\sin\theta)y = 1 \ .$$

The centre of segment AC has coordinates $(u, 0)$ and its right bisector has equation $x = u$. Hence, we find that

$$O \sim \left(u, \frac{1 - u\cos\theta}{\sin\theta}\right)$$

$$N \sim \left(\frac{1}{2}u, \frac{1 - u\cos\theta}{2\sin\theta}\right)$$

$$M \sim (u + \cos\theta, \sin\theta)$$

and

$$D \sim (u + 1, 0) \ .$$

The slope of MD is $(\sin\theta)/(\cos\theta - 1)$. The slope of ND is $(u\cos\theta - 1)/((u+2)\sin\theta)$. Equating these two leads to the equation

$$u(\cos^2\theta - \sin^2\theta - \cos\theta) = 2\sin^2\theta + \cos\theta - 1$$

which reduces to

$$(u + 1)(2\cos^2\theta - \cos\theta - 1) = 0 \ .$$

Since $u + 1 > 0$, we have that $0 = 2\cos^2\theta - \cos\theta - 1 = (2\cos\theta + 1)(\cos\theta - 1)$. Hence $\cos\theta = -1/2$ and so $\angle A = 120°$.

676. Determine all functions f from the set of reals to the set of reals which satisfy the functional equation

$$(x - y)f(x + y) - (x + y)f(x - y) = 4xy(x^2 - y^2)$$

for all real x and y.

Solution. Let u and v be any pair of real numbers. We can solve $x + y = u$ and $x - y = v$ to obtain

$$(x, y) = \left(\frac{1}{2}(u + v), \frac{1}{2}(u - v)\right) \ .$$

From the functional equation, we find that $vf(u) - uf(v) = (u^2 - v^2)uv$, whence

$$\frac{f(u)}{u} - u^2 = \frac{f(v)}{v} - v^2 \ .$$

Thus $(f(x)/x) - x^2$ must be some constant a, so that $f(x) = x^3 + ax$. This checks out for any constant a.

677. For vectors in three-dimensional real space, establish the identity

$$[\mathbf{a} \times (\mathbf{b} - \mathbf{c})]^2 + [\mathbf{b} \times (\mathbf{c} - \mathbf{a})]^2 + [\mathbf{c} \times (\mathbf{a} - \mathbf{b})]^2 = (\mathbf{b} \times \mathbf{c})^2 + (\mathbf{c} \times \mathbf{a})^2 + (\mathbf{a} \times \mathbf{b})^2 + (\mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} + \mathbf{a} \times \mathbf{b})^2 \ .$$

Solution 1. Let $\mathbf{u} = \mathbf{b} \times \mathbf{c}$, $\mathbf{v} = \mathbf{c} \times \mathbf{a}$ and $\mathbf{w} = \mathbf{a} \times \mathbf{b}$. Then, for example, $\mathbf{a} \times (\mathbf{b} - \mathbf{c}) = \mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{c} = \mathbf{a} \times \mathbf{b} + \mathbf{c} \times \mathbf{a} = \mathbf{v} + \mathbf{w}$. The left side is equal to

$$(\mathbf{v}+\mathbf{w})\cdot(\mathbf{v}+\mathbf{w})+(\mathbf{u}+\mathbf{w})\cdot(\mathbf{u}+\mathbf{w})+(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = 2[(\mathbf{u}\cdot\mathbf{u})+(\mathbf{v}\cdot\mathbf{v})+(\mathbf{w}\cdot\mathbf{w})+(\mathbf{u}\cdot\mathbf{v})+(\mathbf{v}\cdot\mathbf{w})+(\mathbf{w}\cdot\mathbf{u})]$$

while the right side is equal to

$$(\mathbf{u}\cdot\mathbf{u}) + (\mathbf{v}\cdot\mathbf{v}) + (\mathbf{w}\cdot\mathbf{w}) + (\mathbf{u}+\mathbf{v}+\mathbf{w})^2$$

which expands to the final expression for the left side.

Solution 2. For vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, we have the identities

$$(\mathbf{u}\times\mathbf{v})\times\mathbf{w} = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{v}\cdot\mathbf{w})\mathbf{u}$$

and

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}\ .$$

Using these, we find for example that

$$\begin{aligned}[\mathbf{a}\times(\mathbf{b}-\mathbf{c})]\cdot[\mathbf{a}\times(\mathbf{b}-\mathbf{c})] &= [\mathbf{a}\times(\mathbf{b}-\mathbf{c})\times\mathbf{a}]\cdot(\mathbf{b}-\mathbf{c})\\ &= \{(\mathbf{a}\cdot\mathbf{a})(\mathbf{b}-\mathbf{c}) - [(\mathbf{b}-\mathbf{c})\cdot\mathbf{a}]\mathbf{a}\}\cdot(\mathbf{b}-\mathbf{c})\\ &= |\mathbf{a}|^2[|\mathbf{b}|^2+|\mathbf{c}|^2-2(\mathbf{b}\cdot\mathbf{c})] - [(\mathbf{b}\cdot\mathbf{a}-\mathbf{c}\cdot\mathbf{a}]^2\\ &= |\mathbf{a}|^2[|\mathbf{b}|^2+|\mathbf{c}|^2-2(\mathbf{b}\cdot\mathbf{c})] - (\mathbf{b}\cdot\mathbf{a})^2 - (\mathbf{c}\cdot\mathbf{a})^2 + 2(\mathbf{b}\cdot\mathbf{a})(\mathbf{c}\cdot\mathbf{a})\ .\end{aligned}$$

Also

$$\begin{aligned}(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{b}\times\mathbf{c}) &= [(\mathbf{b}\cdot\mathbf{b})\mathbf{c} - (\mathbf{c}\cdot\mathbf{b})\mathbf{b}]\cdot\mathbf{c}\\ &= |\mathbf{b}|^2|\mathbf{c}|^2 - (\mathbf{c}\cdot\mathbf{b})^2\end{aligned}$$

and

$$(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{c}\times\mathbf{a}) = [(\mathbf{b}\times\mathbf{c})\times\mathbf{c}]\cdot\mathbf{a} = (\mathbf{b}\cdot\mathbf{c})(\mathbf{c}\cdot\mathbf{a}) - (\mathbf{c}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{a})\ .$$

From these the identity can be checked.

678. For $a, b, c > 0$, prove that

$$\frac{1}{a(b+1)} + \frac{1}{b(c+1)} + \frac{1}{c(a+1)} \geq \frac{3}{1+abc}\ .$$

Solution 1. It is easy to verify the following identity

$$\frac{1}{a(1+b)} + \frac{1}{1+abc} = \frac{1}{1+abc}\left(\frac{1+a}{a(1+b)} + \frac{b(1+c)}{1+b}\right)\ .$$

This and its analogues imply that

$$\frac{1}{a(b+1)} + \frac{1}{b(c+1)} + \frac{1}{c(a+1)} + \frac{3}{1+abc} =$$

$$\frac{1}{1+abc}\left(\frac{1+a}{a(1+b)} + \frac{b(1+c)}{1+b} + \frac{1+b}{b(1+c)} + \frac{c(1+a)}{1+c} + \frac{1+c}{c(1+a)} + \frac{a(1+b)}{1+a}\right)\ .$$

The arithmetic-geometric means inequality yields

$$\frac{1}{a(b+1)} + \frac{1}{b(c+1)} + \frac{1}{c(a+1)} + \frac{3}{1+abc} \geq 6 \times \frac{1}{1+abc}\ .$$

Miraculously, subtracting $3/(1+abc)$ from both sides yields the required inequality. ♡

Solution 2. Multiplying the desired inequality by $(1+abc)a(b+1)b(c+1)c(a+1)$, after some manipulation, produces the equivalent inequality:

$$abc(bc^2+ca^2+ab^2)+(bc+ca+ab)+(abc)^2(a+b+c)+(bc^2+ca^2+ab^2)$$
$$\geq 2abc(a+b+c)+2abc(bc+ca+ab)\ .$$

Pairing off the terms of the left side and applying the arithemetic-geometric means inequality, we get

$$\begin{aligned}(a^2b^3c+bc)+(ab^2c^3+ac)+(a^3bc^2+ab)+(a^3b^2c^2+ab^2)&\\ +(a^2b^3c^2+bc^2)+(a^2b^2c^3+ca^2)&\\ \geq 2ab^2c+2abc^2+2a^2bc+2a^2b^2c+2ab^2c^2+2a^2bc^2&\\ =2abc(a+b+c)+2abc(ab+bc+ca)&\end{aligned}$$

as required.

Solution 3. [C. Deng] Taking the difference between the two sides yields, where the summation is a cyclic one,

$$\begin{aligned}\sum\left(\frac{1}{a(b+1)}-\frac{1}{1+abc}\right)&=\sum\frac{1+abc-a(b+1)}{a(b+1)(1+abc)}\\
&=\frac{1}{1+abc}\sum\left(\frac{b}{b+1}(c-1)-\frac{1}{a(b+1)}(a-1)\right)\\
&=\frac{1}{1+abc}\sum\left(\frac{c}{c+1}(a-1)-\frac{1}{a(b+1)}(a-1)\right)\\
&=\frac{1}{1+abc}\sum(a-1)\left(\frac{c}{c+1}-\frac{1}{a(b+1)}\right)\\
&=\frac{1}{1+abc}\sum\left(\frac{a^2-1}{a}\right)\left(\frac{abc+ac-c-1}{(a+1)(b+1)(c+1)}\right)\\
&=\frac{1}{(1+abc)(1+a)(1+b)(1+c)}\sum\left(a^2bc+a^2c+\frac{c}{a}+\frac{1}{a}-ac-a-bc-c\right)\\
&=\frac{1}{(1+abc)(1+a)(1+b)(1+c)}\sum\left(a^2bc+a^2c-2ab-2a+\frac{b}{c}+\frac{1}{c}\right)\\
&=\frac{1}{(1+abc)(1+a)(1+b)(1+c)}\sum\frac{b+1}{c}(a^2c^2-2ac+1)\\
&=\frac{1}{(1+abc)(1+a)(1+b)(1+c)}\sum\frac{b+1}{c}(ac-1)^2\geq 0\ ,\end{aligned}$$

as desired.

Solution 4. [S. Seraj] Using the Arithmetic-Geometric Means Inequality, we obtain $a^2c+a^2b^2c^3\geq 2a^2bc^2$ and $ab+a^3bc^2\geq 2a^2bc$ and the two cyclic variants of each. Adding the six inequalities yields that

$$a^2c+a^2b^2c^3+ab^2+a^3b^2c^2+bc^2+a^2b^3c^2+ab+a^3bc^2+bc+a^2b^3c+ac+ab^2c^3$$
$$\geq 2a^2bc^2+2a^2b^2c+2ab^2c^2+2a^2bc+2ab^2c+2abc^2\ .$$

Adding the same terms to both sides of the equations, and then factoring the two sides leads to

$$(1+abc)(3abc+a^2bc+ab^2c+abc^2+a^2c+ab^2+bc^2+ab+bc+ca)$$
$$\geq 3abc(abc+ac+bc+ab+a+b+c+1)=3abc(a+1)(b+1)(c+1)\ .$$

Carrying out some divisions and strategically grouping terms in the numerator yields that

$$\frac{(abc^2+bc^2+abc+bc)+(a^2bc+a^2c+abc+ac)+(ab^2c+ab^2+abc+ab)}{abc(a+1)(b+1)(c+1)} \geq \frac{3}{1+abc} \ .$$

Factoring each bracket and simplifying leads to the desired inequality.

679. Let F_1 and F_2 be the foci of an ellipse and P be a point in the plane of the ellipse. Suppose that G_1 and G_2 are points on the ellipse for which PG_1 and PG_2 are tangents to the ellipse. Prove that $\angle F_1PG_1 = \angle F_2PG_2$.

Solution. Let H_1 be the reflection of F_1 in the tangent PG_1, and H_2 be the reflection of F_2 in the tangent PG_2. We have that $PH_1 = PF_1$ and $PF_2 = PH_2$. By the reflection property, $\angle PG_1F_2 = \angle F_1G_1Q = \angle H_1G_1Q$, where Q is a point on PG_1 produced. Therefore, H_1F_2 intersects the ellipse in G_1. Similarly, H_2F_1 intersects the ellipse in K_2. Therefore

$$\begin{aligned} H_1F_2 &= H_1G_1 + G_1F_2 = F_1G_1 + G_1F_2 \\ &= F_1G_2 + G_2F_2 = F_1G_2 + G_2H_2 = H_2F_1 \ . \end{aligned}$$

Therefore, triangle PH_1F_2 and PF_1H_2 are congruent (SSS), so that $\angle H_1PF_2 = \angle H_2PF_1$. It follows that

$$2\angle F_1PG_1 = \angle H_1PF_1 = \angle H_2PF_2 = 2\angle F_2PG_2$$

and the desired result follows.

680. Let $u_0 = 1$, $u_1 = 2$ and $u_{n+1} = 2u_n + u_{n-1}$ for $n \geq 1$. Prove that, for every nonnegative integer n,

$$u_n = \sum \left\{ \frac{(i+j+k)!}{i!j!k!} : i,j,k \geq 0, i+j+2k = n \right\} \ .$$

Solution 1. Suppose that we have a supply of white and of blue coaches, each of length 1, and of red coaches, each of length 2; the coaches of each colour are indistinguishable. Let v_n be the number of trains of total length n that can be made up of red, white and blue coaches of total length n. Then $v_0 = 1$, $v_1 = 2$ and $v_2 = 5$ (R, WW, WB, BW, BB). In general, for $n \geq 1$, we can get a train of length $n+1$ by appending either a white or a blue coach to a train of length n or a red coach to a train of length $n-1$, so that $v_{n+1} = 2v_n + v_{n-1}$. Therefore $v_n = u_n$ for $n \geq 0$.

We can count v_n in another way. Suppose that the train consists of i white coaches, j blue coaches and k red coaches, so that $i+j+2k = n$. There are $(i+j+k)!$ ways of arranging the coaches in order; any permutation of the i white coaches among themselves, the j blue coaches among themselves and k red coaches among themselves does not change the train. Therefore

$$u_n = \sum \left\{ \frac{(i+j+k)!}{i!j!k!} : i,j,k \geq 0, i+j+2k = n \right\} \ .$$

Solution 2. Let $f(t) = \sum_{n=0}^{\infty} u_n t^n$. Then

$$\begin{aligned} f(t) &= u_0 + u_1 t + (2u_1 + u_0)t^2 + (2u_2 + u_1)t^3 + \cdots \\ &= u_0 + u_1 t + 2t(f(t) - u_0) + t^2 f(t) = u_0 + (u_1 - 2u_0)t + (2t + t^2) f(t) \\ &= 1 + (2t + t^2) f(t) \ , \end{aligned}$$

whence

$$\begin{aligned} f(t) &= \frac{1}{1-2t-t^2} = \frac{1}{1-t-t-t^2} \\ &= \sum_{n=0}^{\infty} (t+t+t^2)^n = \sum_{n=0}^{\infty} t^n \left[\sum \left\{ \frac{(i+j+k)!}{i!j!k!} : i,j,k \geq 0, i+j+2k = n \right\} \right] \ . \end{aligned}$$

Solution 3. Let w_n be the sum in the problem. It is straightforward to check that $u_0 = w_0$ and $u_1 = w_1$. We show that, for $n \geq 1$, $w_{n+1} = 2w_n + w_{n-1}$ from which it follows by induction that $u_n = w_n$ for each n. By convention, let $(-1)! = \infty$. Then, for $i, j, k \geq 0$ and $i + j + 2k = n + 1$, we have that

$$\begin{aligned}\frac{(i+j+k)!}{i!j!k!} &= \frac{(i+j+k)(i+j+k-1)!}{i!j!k!} \\ &= \frac{(i+j+k-1)!}{(i-1)!j!k!} + \frac{(i+j+k-1)!}{i!(j-1)!k!} + \frac{(i+j+k-1)!}{i!j!(k-1)!} \ ,\end{aligned}$$

whence

$$\begin{aligned}w_{n+1} = &\sum \left\{ \frac{(i+j+k-1)!}{(i-1)!j!k!} : i, j, k \geq 0, (i-1) + j + 2k = n \right\} \\ &+ \sum \left\{ \frac{(i+j+k-1)!}{i!(j-1)!k!} : i, j, k \geq 0, i + (j-1) + 2k = n \right\} \\ &+ \sum \left\{ \frac{(i+j+k-1)!}{i!j!(k-1)!} : i, j, k \geq 0, i + j + 2(k-1) = n - 1 \right\} \\ = &\ w_n + w_n + w_{n-1} = 2w_n + w_{n-1}\end{aligned}$$

as desired.

681. Let $\mathbf{a}$ and $\mathbf{b}$, the latter nonzero, be vectors in $\mathbf{R}^3$. Determine the value of λ for which the vector equation

$$\mathbf{a} - (\mathbf{x} \times \mathbf{b}) = \lambda \mathbf{b}$$

is solvable, and then solve it.

Solution 1. If there is a solution, we must have $\mathbf{a} \cdot \mathbf{b} = \lambda |\mathbf{b}|^2$, so that $\lambda = (\mathbf{a} \cdot \mathbf{b})/|\mathbf{b}|^2$. On the other hand, suppose that λ has this value. Then

$$\begin{aligned}\mathbf{0} &= \mathbf{b} \times \mathbf{a} - \mathbf{b} \times (\mathbf{x} \times \mathbf{b}) \\ &= \mathbf{b} \times \mathbf{a} - [(\mathbf{b} \cdot \mathbf{b})\mathbf{x} - (\mathbf{b} \cdot \mathbf{x})\mathbf{b}]\end{aligned}$$

so that

$$\mathbf{b} \times \mathbf{a} = |\mathbf{b}|^2 \mathbf{x} - (\mathbf{b} \cdot \mathbf{x})\mathbf{b} \ .$$

A particular solution of this equation is

$$\mathbf{x} = \mathbf{u} \equiv \frac{\mathbf{b} \times \mathbf{a}}{|\mathbf{b}|^2} \ .$$

Let $\mathbf{x} = \mathbf{z}$ be any other solution. Then

$$\begin{aligned}|\mathbf{b}|^2(\mathbf{z} - \mathbf{u}) &= |\mathbf{b}|^2 \mathbf{z} - |\mathbf{b}|^2 \mathbf{u} \\ &= (\mathbf{b} \times \mathbf{a} + (\mathbf{b} \cdot \mathbf{z})\mathbf{b}) - (\mathbf{b} \times \mathbf{a} + (\mathbf{b} \cdot \mathbf{u})\mathbf{b}) \\ &= (\mathbf{b} \cdot \mathbf{z})\mathbf{b}\end{aligned}$$

so that $\mathbf{z} - \mathbf{u} = \mu \mathbf{b}$ for some scalar μ.

We check when this works. Let $\mathbf{x} = \mathbf{u} + \mu \mathbf{b}$ for some scalar μ. Then

$$\begin{aligned}\mathbf{a} - (\mathbf{x} \times \mathbf{b}) = \mathbf{a} - (\mathbf{u} \times \mathbf{b}) &= \mathbf{a} - \frac{(\mathbf{b} \times \mathbf{a}) \times \mathbf{b}}{|\mathbf{b}|^2} \\ &= \mathbf{a} + \frac{\mathbf{b} \times (\mathbf{b} \times \mathbf{a})}{|\mathbf{b}|^2} \\ &= \mathbf{a} + \frac{(\mathbf{b} \cdot \mathbf{a})\mathbf{b} - (\mathbf{b} \cdot \mathbf{b})\mathbf{a}}{|\mathbf{b}|^2} \\ &= \mathbf{a} + \left(\frac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{b}|^2} \right) \mathbf{b} - \mathbf{a} = \lambda \mathbf{b} \ ,\end{aligned}$$

as desired. Hence, the solutions is

$$\mathbf{x} = \frac{\mathbf{b} \times \mathbf{a}}{|\mathbf{b}|^2} + \mu \mathbf{b} \ ,$$

where μ is an arbitrary scalar.

Solution 2. [B. Yahagni] Suppose, to begin with, that $\{\mathbf{a}, \mathbf{b}\}$ is linearly dependent. Then $\mathbf{a} = [(\mathbf{a} \cdot \mathbf{b})/|\mathbf{b}|^2]\mathbf{b}$. Since $(\mathbf{x} \times \mathbf{b}) \cdot \mathbf{b} = 0$ for all $\mathbf{x}$, the equation has no solutions except when $\lambda = (\mathbf{a} \cdot \mathbf{b})/|\mathbf{b}|^2$. In this case, it becomes $\mathbf{x} \times \mathbf{b} = \mathbf{0}$ and is satisfied by $\mathbf{x} = \mu\mathbf{b}$, where μ is any scalar.

Otherwise, $\{\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b}\}$ is linearly independent and constitutes a basis for $\mathbb{R}^3$. Let a solution be

$$\mathbf{x} = \alpha\mathbf{a} + \mu\mathbf{b} + \beta(\mathbf{a} \times \mathbf{b}) \ .$$

Then

$$\mathbf{x} \times \mathbf{b} = \alpha(\mathbf{a} \times \mathbf{b}) + \beta[(\mathbf{a} \times \mathbf{b}) \times \mathbf{b}] = \alpha(\mathbf{a} \times \mathbf{b}) + \beta(\mathbf{a} \cdot \mathbf{b})\mathbf{b} - \beta(\mathbf{b} \cdot \mathbf{b})\mathbf{a}$$

and the equation becomes

$$(1 + \beta|\mathbf{b}|^2)\mathbf{a} - \beta(\mathbf{a} \cdot \mathbf{b})\mathbf{b} - \alpha(\mathbf{a} \times \mathbf{b}) = \lambda\mathbf{b} \ .$$

Therefore $\alpha = 0$, μ is arbitrary, $\beta = -1/|\mathbf{b}|^2$ and $\lambda = -\beta(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \mathbf{b})/|\mathbf{b}|^2$.

Therefore, the existence of a solution requires that $\lambda = (\mathbf{a} \cdot \mathbf{b})/|\mathbf{b}|^2$ and the solution then is

$$\mathbf{x} = \mu\mathbf{b} - \frac{1}{|\mathbf{b}|^2}(\mathbf{a} \times \mathbf{b}) \ .$$

Solution 3. Writing the equation in vector components yields the system

$$b_3x_2 - b_2x_3 = a_1 - \lambda b_1 \ ;$$

$$-b_3x_1 + b_1x_3 = a_2 - \lambda b_2 \ ;$$

$$b_2x_1 - b_1x_2 = a_3 - \lambda b_3 \ .$$

The matrix of coefficients of the left side is of rank 2, so that the corresponding homogeneous system of equations has a single infinity of solutions. Multiplying the three equations by b_1, b_2 and b_3 respectively and adding yields

$$0 = a_1b_1 + a_2b_2 + a_3b_3 - \lambda(b_1^2 + b_2^2 + b_3^2) \ .$$

Thus, for a solution to exist, we require that

$$\lambda = \frac{a_1b_1 + a_2b_2 + a_3b_3}{b_1^2 + b_2^2 + b_3^2)} .$$

In addition, we learn that the corresponding homogeneous system is satisfied by

$$(x_1, x_2, x_3) = \mu(b_1, b_2, b_3)$$

where μ is an arbitrary scalar.

It remains to find a particular solution for the nonhomogeneous system. Multiplying the third equation by b_2 and subtracting the second multiplied by b_3, we obtain that

$$(b_2^2 + b_3^2)x_1 = b_1(b_2x_2 + b_3x_3) + (a_3b_2 - a_2b_3) \ .$$

Therefore, setting $b_1^2 + b_2^2 + b_3^2 = b^2$, we have that

$$b^2x_1 = b_1(b_1x_1 + b_2x_2 + b_3x_3) + (a_3b_2 - a_2b_3) \ .$$

Similarly

$$b^2x_2 = b_2(b_1x_1 + b_2x_2 + b_3x_3) + (a_1b_3 - a_3b_1) \ ,$$

$$b^2x_3 = b_3(b_1x_1 + b_2x_2 + b_3x_3) + (a_2b_1 - a_1b_2) \ .$$

Observing that $b_1x_1 + b_2x_2 + b_3x_3$ vanishes when

$$(x_1, x_2, x_3) = (a_3b_2 - a_2b_3, a_1b_3 - a_3b_1, a_2b_1 - a_1b_2) \ ,$$

we obtain a particular solution to the system:

$$(x_1, x_2, x_3) = b^{-2}(a_3b_2 - a_2b_3, a_1b_3 - a_3b_1, a_2b_1 - a_1b_2) \ .$$

Adding to this the general solution of the homogeneous system yields the solution of the nonhomogeneous system.

682. The plane is partitioned into n regions by three families of parallel lines. What is the least number of lines to ensure that $n \geq 2010$?

Solution. Suppose that there are x, y and z lines in the three families. Assume that no point is common to three distinct lines. The $x+y$ lines of the first two families partition the plane into $(x+1)(y+1)$ regions. Let λ be one of the lines of the third family. It is cut into $x+y+1$ parts by the lines in the first two families, so the number of regions is increased by $x+y+1$. Since this happens z times, the number of regions that the plane is partitioned into by the three families of

$$n = (x+1)(y+1) + z(x+y+1) = (x+y+z) + (xy+yz+zx) + 1 \ .$$

Let $u = x+y+z$ and $v = xy+yz+zx$. Then (by the Cauchy-Schwarz Inequality for example), $v \leq x^2+y^2+z^2$, so that $u^2 = x^2+y^2+z^2+2v \geq 3v$. Therefore, $n \leq u + \frac{1}{3}u^2 + 1$. This takes the value 2002 when $u = 76$. However, when $(x,y,z) = (26,26,25)$, then $u = 77$, $v = 1976$ and $n = 2044$. Therefore, we need at least 77 lines, but a suitably chosen set of 77 lines will suffice.

683. Let $f(x)$ be a quadratic polynomial. Prove that there exist quadratic polynomials $g(x)$ and $h(x)$ for which

$$f(x)f(x+1) = g(h(x)) \ ,$$

Solution 1. [A. Remorov] Let $f(x) = a(x-r)(x-s)$. Then

$$\begin{aligned} f(x)f(x+1) &= a^2(x-r)(x-s+1)(x-r+1)(x-s) \\ &= a^2(x^2+x-rx-sx+rs-r)(x^2+x-rx-sx+rs-s) \\ &= a^2[(x^2-(r+s-1)x+rs)-r][(x^2-(r+s-1)x+rs)-s] \\ &= g(h(x)) \ , \end{aligned}$$

where $g(x) = a^2(x-r)(x-s) = af(x)$ and $h(x) = x^2-(r+s-1)x+rs$.

Solution 2. Let $f(x) = ax^2+bx+c$, $g(x) = px^2+qx+r$ and $h(x) = ux^2+vx+w$. Then

$$f(x)f(x+1) = a^2x^4 + 2a(a+b)x^3 + (a^2+b^2+3ab+2ac)x^2 + (b+2c)(a+b)x + c(a+b-c)$$

$$\begin{aligned} g(h(x)) &= p(ux^2+vx+w)^2 + q(ux+vx+w) + r \\ &= pu^2x^4 + 2puvx^3 + (2puw+pv^2+qu)x^2 + (2pvw+qv)x + (pw^2+qw+r) \ . \end{aligned}$$

Equating coefficients, we find that $pu^2 = a^2$, $puv = a(a+b)$, $2puw+pv^2+qu = a^2+b^2+3ab+2ac$, $(b+2c)(a+b) = (2pw+q)v$ and $c(a+b+c) = pw^2+qw+r$. We need to find just one solution of this system. Let $p = 1$ and $u = a$. Then $v = a + b$ and $b + 2c = 2pw + q$ from the second and fourth equations. This yields the third equation automatically. Let $q = b$ and $w = c$. Then from the fifth equation, we find that $r = ac$.

Thus, when $f(x) = ax^2 + bx + c$, we can take $g(x) = x^2 + bx + ac$ and $h(x) = ax^2 + (a + b)x + c$.

Solution 3. [S. Wang] Suppose that

$$f(x) = a(x + h)^2 + k = a(t - (1/2))^2 + k \ ,$$

where $t = x + h + \frac{1}{2}$. Then $f(x + 1) = a(x + 1 + h)^2 + k = a(t + (1/2))^2 + k$, so that

$$\begin{aligned} f(x)f(x+1) &= a^2(t^2 - (1/4))^2 + 2ak(t^2 + (1/4)) + k^2 \\ &= a^2t^4 + \left(-\frac{a^2}{2} + 2ak\right)t^2 + \left(\frac{a^2}{16} + \frac{ak}{2} + k^2\right) . \end{aligned}$$

Thus, we can achieve the desired representation with $h(x) = t^2 = x^2 + (2h + 1)x + \frac{1}{4}$ and $g(x) = a^2x^2 + (\frac{-a^2}{2} + 2ak)x + (\frac{a^2}{16} + \frac{ak}{2} + k^2)$.

Solution 4. [V. Krakovna] Let $f(x) = ax^2 + bx + c = au(x)$ where $u(x) = x^2 + dx + e$, where $b = ad$ and $c = ae$. If we can find functions $v(x)$ and $w(x)$ for which $u(x)u(x + 1) = v(w(x))$, then $f(x)f(x + 1) = a^2v(w(x))$, and we can take $h(x) = w(x)$ and $g(x) = a^2v(x)$.

Define $p(t) = u(x + t)$, so that $p(t)$ is a monic quadratic in t. Then, noting that $p''(t) = u''(x + t) = 2$, we have that

$$p(t) = u(x + t) = u(x) + u'(x)t + \frac{u''(x)}{2}t^2 = t^2 + u'(x)t + u(x) \ ,$$

from which we find that

$$\begin{aligned} u(x)u(x+1) &= p(0)p(1) = u(x)[u(x) + u'(x) + 1] \\ &= u(x)^2 + u'(x)u(x) + u(x) = p(u(x)) = u(x + u(x)) \ . \end{aligned}$$

Thus, $u(x)u(x + 1) = v(w(x))$ where $w(x) = x + u(x)$ and $v(x) = u(x)$. Therefore, we get the desired representation with

$$h(x) = x + u(x) = x^2 + \left(1 + \frac{b}{a}\right)x + \frac{c}{a}$$

and

$$g(x) = a^2v(x) = a^2u(x) = af(x) = a^2x^2 + abx + ac \ .$$

Solution 5. [Generalization by J. Rickards.] The following statement is true: *Let the quartic polynomial $f(x)$ have roots r_1, r_2, r_3, r_4 (not necessarily distinct). Then $f(x)$ can be expressed in the form $g(h(x)$ for quadratic polynomials $g(x)$ and $h(x)$ if and only if the sum of two of r_1, r_2, r_3, r_4 is equal to the sum of the other two.*

Wolog, suppose that $r_1 + r_2 = r_3 + r_4$. Let the leading coefficient of $f(x)$ be a. Define $h(x) = (x - r_1)(x - r_2)$ and $g(x) = a(x - r_3^2 + r_1r_3 + r_2r_3 - r_1r_2)$. Then

$$\begin{aligned} g(h(x) &= a(x - r_1)(x - r_2)[(x - r_1)(x - r_2) - r_3^2 + r_1r - 3 + r_2r_3 - r_1r_2 \\ &= a(x - r_1)(x - r_2)[x^2 - (r_1 + r_2)x - r_3^2 + r_1r_3 + r_2r_3) \\ &= a(x - r_1)(x - r_2)[x^2 - (r_3 + r - 4)x + r_3(r_1 + r_2 - r_3)] \\ &= a(x - r_1)(x - r_2)(x^2 - (r - 3 + r_4)x + r - 3r_4 \\ &= a(x - r_1)(x - r_2)(x - r_3)(x - r_4) \end{aligned}$$

as required.

Conversely, assume that we are given quadratic polynomials $g(x) = b(x-r_5)(x-r_6)$ and $h(x)$ and that c is the leading coefficient of $h(x)$. Let $f(x) = g(h(x))$.

Suppose that

$$h(x) - r_5 = c(x-r_1)(x-r_2)$$

and that

$$h(x) - r_6 = c(x-r_3)(x-r_4) \ .$$

Then

$$f(x) = g(h(x)) = bc^2(x-r_1)(x-r_2)(x-r_3)(x-r_4) \ .$$

We have that

$$h(x) = c(x-r_1)(x-r_2) + r_5 = cx62 - c(r_1+r_2)x + cr_1r_2 + r_5$$

and

$$h(x) = c9x - r_3)(x-r_4) + r_6 = cx^2 - c(r_3+r_4)x + cr_3r_4 + r_6 \ ,$$

whereupon it follows that $r_1 + r_2 = r_3 + r_4$ and the desired result follows.

Comment. The second solution can also be obtained by looking at special cases, such as when $a = 1$ or $b = 0$, getting the answer and then making a conjecture.