2020 CMO Qualifying Repêchage

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Official Problem Set

1. Show that for all integers $a \geq 1$, $\lfloor \sqrt{a} + \sqrt{a+1} + \sqrt{a+2} \rfloor = \lfloor \sqrt{9a+8} \rfloor$.

2. Given a set S, of integers, an *optimal partition of* S into sets T, U is a partition which minimizes the value $|t-u|$, where t and u are the sum of the elements of T and U respectively.

   Let P be a set of distinct positive integers such that the sum of the elements of P is $2k$ for a positive integer k, and no subset of P sums to k.

   Either show that there exists such a P with at least 2020 different optimal partitions, or show that such a P does not exist.

3. Let N be a positive integer and $A = a_1, a_2, \ldots, a_N$ be a sequence of real numbers. Define the sequence $f(A)$ to be

$$f(A) = \left(\frac{a_1 + a_2}{2}, \frac{a_2 + a_3}{2}, \cdots, \frac{a_{N-1} + a_N}{2}, \frac{a_N + a_1}{2} \right)$$

   and for k a positive integer define $f^k(A)$ to be f applied to A consecutively k times (i.e. $f(f(\cdots f(A))))$)

   Find all sequences $A = (a_1, a_2, \ldots, a_N)$ of integers such that $f^k(A)$ contains only integers for all k.

4. Determine all graphs G with the following two properties:

   - G contains at least one Hamilton path.
   - For any pair of vertices, $u, v \in G$, if there is a Hamilton path from u to v then the edge uv is in the graph G.

5. We define the following sequences:

    - Sequence A has $a_n = n$.
    - Sequence B has $b_n = a_n$ when $a_n \not\equiv 0 \pmod 3$ and $b_n = 0$ otherwise.
    - Sequence C has $c_n = \sum_{i=1}^{n} b_i$.
    - Sequence D has $d_n = c_n$ when $c_n \not\equiv 0 \pmod 3$ and $d_n = 0$ otherwise.
    - Sequence E has $e_n = \sum_{i=1}^{n} d_i$.

    Prove that the terms of sequence E are exactly the perfect cubes.

6. In convex pentagon $ABCDE$, AC is parallel to DE, AB is perpendicular to AE, and BC is perpendicular to CD. If H is the orthocentre of triangle ABC and M is the midpoint of segment DE, prove that AD, CE and HM are concurrent.

7. Let a, b, c be positive real numbers with $ab + bc + ac = abc$. Prove that

$$\frac{bc}{a^{a+1}} + \frac{ac}{b^{b+1}} + \frac{ab}{c^{c+1}} \geq \frac{1}{3}.$$

8. Find all pairs (a, b) of positive rational numbers such that $\sqrt[b]{a} = ab$.