Crux Mathematicorum

VOLUME 38, NO. 6

JUNE / JUIN 2012

IN THIS ISSUE / DANS CE NUMÉRO

Published by
Canadian Mathematical Society
209 - 1725 St. Laurent Blvd.
Ottawa, Ontario, Canada K1G 3V4
FAX: 613–733–8994
email: subscriptions@cms.math.ca

Publié par
Société mathématique du Canada
209 - 1725 boul. St. Laurent
Ottawa (Ontario) Canada K1G 3V4
Téléc : 613–733–8994
email : abonnements@smc.math.ca

EDITORIAL

Shawn Godin

We have been working on a few new features this volume. The *Contest Corner* started earlier this year featuring problems that have appeared in mathematics contests at the high school and undergraduate levels. We are starting to receive solutions to these problems and we look forward to more solutions from our readers. The solutions will start to appear next volume.

Last issue saw the first *Department Highlight*. These will appear every second issue and give our readers some insight into the programs and activities at various mathematics departments across Canada.

Last issue we also re-launched the *Problem of the Month*. A column of the same name appeared in the *Mathematical Mayhem* section for years and presented the discussion of a problem from a high school mathematics contest. With the re-launch, we are keeping the same format, yet we will be including any problem that would be of interest to readers of ***Crux Mathematicorum***. This column will be dedicated to the former Editor-in-Chief of ***Crux Mathematicorum*** and my personal friend and mentor, the late Jim Totten. Jim had a great love of mathematics and shared that love with his students and others through his problem of the week, his work on mathematics contests and outreach programs. We will feature problems in this section that we hope Jim would have enjoyed.

This issue we introduce a new column, the *Problem Solver's Toolkit*. This will be similar to Michel Bataille's *Focus On ...*, which also started to appear this volume, although it will be a bit more elementary in nature. *Problem Solver's Toolkit* will feature techniques that will be of use to mathematical problem solvers at all levels.

We hope that you enjoy our new problem column and other features. We always welcome your feedback.

Shawn Godin

MAYHEM SOLUTIONS

Mathematical Mayhem *is being reformatted as a stand-alone mathematics journal for high school students. Solutions to problems that appeared in the last volume of* **Crux** *will appear in this volume, after which time* **Mathematical Mayhem** *will be discontinued in* **Crux**. *New* **Mayhem** *problems will appear when the journal is relaunched in 2013.*

M495. *Proposed by the Mayhem Staff.*

All possible lines are drawn through the point $(0,0)$ and the points (x,y), where x and y are whole numbers with $1 \leq x, y \leq 10$. How many distinct lines are drawn?

Solution by Florencio Cano Vargas, Inca, Spain.

Since the lines pass by $(0,0)$, each line is characterized by a single parameter: the slope m and counting the lines amounts to counting all possible values of the slope. Since $m = \frac{y}{x}$, the different values of the slope are the irreducible fractions $\frac{y}{x}$ with $1 \leq x, y \leq 10$. The boundaries are $\frac{1}{10} \leq m \leq \frac{10}{1}$.

Let us call N the number of lines, $N[m<1]$ the number of lines with $m < 1$ and $N[m>1]$ the number of lines with $m > 1$. By symmetry around the line $x = y$ (i.e. $m = 1$), we have $N[m<1] = N[m>1]$ and the requested number can be written as:

$$N = 2N[m<1] + 1$$

where the last term accounts for the case $m = 1$ which is considered separately.

To evaluate $N[m<1]$ we still have to count the number of irreducible fractions $\frac{y}{x}$ with $1 \leq y < x \leq 10$. We study the different cases for x:

$x = 1$. This case gives no values of $m < 1$.

$x = 2$. We look for irreducible fractions $\frac{y}{2}$, i.e., values of y relatively prime to 2, which is just $y = 1$.

$x = 3$. We look for irreducible fractions $\frac{y}{3}$, i.e., values of y relatively prime to 3, which are two values $y = 1, 2$.

From these cases it can be inferred that we look for the number of values of y relatively prime to x and which are smaller than x. This is just Euler's totient function $\varphi(x)$ (Euler's totient function $\varphi(n)$ is defined as the number of positive integers less than or equal to n that are relatively prime to n. We are looking for the number of positive integers strictly less than n which are relatively prime to n, but this subtlety makes no difference since a number is not relatively prime to itself.) Then we can write:

$$N[m<1] = \sum_{x=2}^{10} \varphi(x) = 1 + 2 + 2 + 4 + 2 + 6 + 4 + 6 + 4 = 31$$

and the final answer is $N = 2 \cdot 31 + 1 = 63$ different lines.

Also solved by KONSTANTINOS DAGIADAS, Agrinio, Greece; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; CÁSSIO DOS SANTOS SOUSA, Instituto Tecnológico de Aeronáutica, São Paulo, Brazil; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA.

M496. *Proposed by Sally Li, student, Marc Garneau Collegiate Institute, Toronto, ON.*

Show that if we write the numbers from 1 to n around a circle, in any order, then, for all $x = 1, 2, \dots, n$, we are guaranteed to find a block of x consecutive numbers that add up to at least $\left\lceil \dfrac{x(n+1)}{2} \right\rceil$. Here $\lceil y \rceil$ is the ceiling function, that is, the least integer greater than or equal to y. So $\lceil 6.2 \rceil = 7$, $\lceil \pi \rceil = 4$, $\lceil -8.3 \rceil = -8$ and $\lceil 10 \rceil = 10$.

Solution by Florencio Cano Vargas, Inca, Spain.

Let us first note that the sum of all the numbers of the circle is given by $\frac{n(n+1)}{2}$.

Let $x < n$ be the number of consecutive numbers we take at a time. For a given x we have n different combinations of consecutive numbers and each number of the circle enters in x combinations. If we define S_k as the sum of the numbers in the k^{th} combination ($k = 1, \dots, n$), then the sum of the numbers contained in all the combinations $\{S_1, S_2, \dots, S_n\}$ is x times the sum of the numbers of the circle:

$$S_1 + S_2 + \cdots + S_n = x \cdot \frac{n(n+1)}{2} \tag{1}$$

hence

$$\frac{S_1 + S_2 + \cdots + S_n}{n} = \frac{x(n+1)}{2}.$$

Then the problem is equivalent to prove that in an arithmetic mean of positive integers, there is at least one which is larger than or equal to the mean. Let us suppose that it is not so, and that all S_k satisfy:

$$S_k < \frac{x(n+1)}{2}, \quad k = 1, \dots, n,$$

but then

$$S_1 + S_2 + \cdots + S_n < x \cdot \frac{n(n+1)}{2},$$

which contradicts (1). Therefore, there must be at least some $S_k \geq \frac{x(n+1)}{2}$. Since S_k is integer for any k, but $\frac{x(n+1)}{2}$ isn't necessarily an integer, then we can strengthen the claim, that is, there will be at least one k for which

$$S_k \geq \left\lceil \frac{x(n+1)}{2} \right\rceil .$$

It remains to prove the case $x = n$. In that case, the only block is the whole set of numbers, whose sum is $\frac{n(n+1)}{2} = \left\lceil \frac{x(n+1)}{2} \right\rceil$, which completes the proof.

Also solved by the proposer.

M497. *Proposed by Pedro Henrique O. Pantoja, student, UFRN, Brazil.*

Find all integers a, b, c where c is a prime number such that a^b+c and a^b-c are both perfect squares.

Solution by David E. Manes, SUNY at Oneonta, Oneonta, NY, USA.

Assume c is an odd prime and $a^b+c=m^2$ and $a^b-c=n^2$ for some integers m and n. Subtracting the two equations yields $2c=m^2-n^2=(m+n)(m-n)$. Unique factorization then implies $m-n=2$ and $m+n=c$, which are contradictory equations since $m \pm n$ always have the same parity. Hence, if c is an odd prime, there are no integers a, b such that a^b+c and a^b-c are both perfect squares.

However, if $c=2$, then $a^b+c=m^2$ and $a^b-c=n^2$ imply $m^2-n^2=(m+n)(m-n)=4$. Therefore, either $m+n=4$ and $m-n=1$ or $m+n=m-n=2$ by unique factorization. As noted above, the equations $m+n=4$ and $m-n=1$ are contradictory. If $m+n=m-n=2$, then $m=2$ and $n=0$. Therefore $a^b=2$, whence $a=2$ and $b=1$. Thus, $(a,b,c)=(2,1,2)$ is the unique solution.

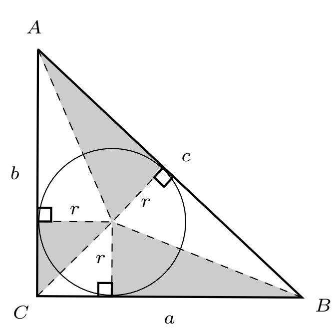
Also solved by the proposer.

M498. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Right triangle ABC has its right angle at C. The two sides CB and CA are of integer length. Determine the condition for the radius of the incircle of triangle ABC to be a rational number.

Solution by Cássio dos Santos Sousa, Instituto Tecnológico de Aeronáutica, São Paulo, Brazil.

Take $BC=a$, $AC=b$, and $AB=c$. To calculate the radius r of the incircle of triangle ABC, we may use the figure below.



As the incenter is the intersection of the angle bisectors, there will be three pairs of congruent triangles formed (one shaded from each pair), each can be proved using AAS comparison (one of the angles comes from the bisection, the other angle is $90°$, and the common side has length r). Then, from the triangles formed with vertex A:

$$b - r = c - (a - r)$$
$$r = \frac{a+b-c}{2}.$$

We were given that a and b are integers, hence, if we want a rational value for r, then c must be rational. The Pythagorean theorem gives $c^2 = a^2 + b^2$, an integer, so c is either integer or irrational. If we want r to be rational, then c must be an integer.

To find a and b such that $\sqrt{a^2+b^2}$ will be an integer value, we may use Euclid's Formula:

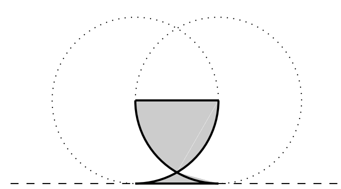
$$a = k(m^2 - n^2)$$
$$b = k(2mn)$$
$$c = k(m^2 + n^2)$$

If k, m and n are integers, and $m > n$, then (a, b, c) will be a Pythagorean triple, and then r will be a rational value.

Also solved by KONSTANTINOS DAGIADAS, Agrinio, Greece; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; CÁSSIO DOS SANTOS SOUSA, Instituto Tecnológico de Aeronáutica, São Paulo, Brazil; KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA; and the proposer.

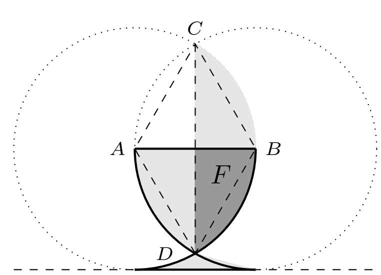
M499. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Two circles of radius 1 are drawn so that each circle passes through the centre of the other circle. Find the area of the goblet like region contained between the common radius, the circumferences and one of the common tangents as shown in the diagram to the right.



Solution by Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany.

Let A, B, C, and D be the points as shown in the figure. Let F be the darkly shaded area, that is, half of the upper part of the goblet. Triangles ABC and ABD are equilateral, since each side is a radius of one of the circles. By symmetry, the second half of the upper part of the goblet, the part beside F, is equal to the shaded area above F.

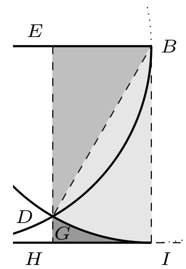


Hence, the area of the upper part of the goblet, $2F$, is equal to the area of a 120° sector, minus the area of triangle ACD.

Note that triangles AEC, BEC, AED and BED are congruent by hypotenuse-side, hence the $[ACD] = [AEC] + [AED] = [AEC] + [BEC] = [ABC]$ (where $[XYZ]$ represents the area of triangle XYZ). Thus

$$2F = \frac{\pi}{3} - \frac{\sqrt{3}}{4}.$$

Let E, H and I be the points as shown in the diagram. Let G be the darkest shaded area, that is, half of the lower part of the goblet. Looking at the rectangle $BEHI$ with sides 1 and $\frac{1}{2}$, it is broken into three parts: half of the base of the goblet (G), triangle BED and sector BDI. The triangle is half of an equilateral triangle, and the sector has angle 30°, hence



$$G = \frac{1}{2} - \frac{\sqrt{3}}{8} - \frac{\pi}{12}.$$

Thus the area of the goblet is

$$2F + 2G = \frac{\pi}{3} - \frac{\sqrt{3}}{4} + 2\left(\frac{1}{2} - \frac{\sqrt{3}}{8} - \frac{\pi}{12}\right) = 1 + \frac{\pi}{6} - \frac{\sqrt{3}}{2} \doteq 0.6576.$$

Also solved by FLORENCIO CANO VARGAS, Inca, Spain; ANDHIKA GILANG, student, SMPN 8, Yogyakarta, Indonesia; THARIQ SURYA GUMELAR, student, SMPN 8, Yogyakarta, Indonesia; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; ALMER FANDRIYANTO, student, SMAN 25, Bandung, Indonesia; KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA; and the proposer. One correct solution with no name on it and one incorrect solution were also received.

M500. *Proposed by Edward T.H. Wang and Dexter S.Y. Wei, Wilfrid Laurier University, Waterloo, ON.*

Let $\mathbb{N}$ denote the set of natural numbers.

(a) Show that if $n \in \mathbb{N}$, there do not exist $a, b \in \mathbb{N}$ such that $\dfrac{[a,b]}{a+b} = n$, where $[a,b]$ denotes the least common multiple of a and b.

(b) Show that for any $n \in \mathbb{N}$, there exist infinitely many triples (a, b, c) of natural numbers such that $\dfrac{[a,b,c]}{a+b+c} = n$, where $[a,b,c]$ denotes the least common multiple of a, b and c.

Solution by Florencio Cano Vargas, Inca, Spain.

(a) Let us suppose that for a given $n \in \mathbb{N}$, there exist $a, b \in \mathbb{N}$ that satisfy the condition given in the problem. We can write $a = da', b = db'$ where $d = gcd(a,b)$. Then $\gcd(a', b') = 1$ and $[a,b] = da'b'$ so the condition of the problem can be rewritten as:

$$\frac{a'b'}{a'+b'} = n.$$

First of all note that since a' and b' are relatively prime and we cannot have $a'+b'=1$, then $a'+b'\neq 1$ must divide $a'b'$.

Let $p>1$ be any prime common factor of $a'b'$ and $a'+b'$. Since a' and b' are relatively prime they don't share any prime factor, and therefore p is a factor either of a' or b'. Let us assume without loss of generality that it is a factor of a', i.e. $a'=ps$ for some integer s. Then for some integer q

$$a'+b'=pq \Rightarrow ps+b'=pq \Rightarrow b'=p(q-s)$$

and then p is also a factor of b' which contradicts the fact that $\gcd(a',b')=1$. This means that $a'b'$ and $(a'+b')$ are relatively prime and then the fraction $\frac{a'b'}{a'+b'}$ is irreducible and $n\notin \mathbb{N}$, which contradicts the initial assumption.

(b) We can look for triples (a,b,c) such that $a=a'd, b=b'd, c=c'd$ with $\gcd(a,b,c)=d$ and with a',b',c' pairwise relatively prime. Then we can write $\operatorname{lcm}(a,b,c)=da'b'c'$ and the condition of the problem can be rewritten as:

$$\frac{a'b'c'}{a'+b'+c'}=n.$$

To enforce this property let us choose $b'=a'+1$, which is always relatively prime with a' and $c'=1$. We end up with a condition for a':

$$\frac{a'(a'+1)}{a'+(a'+1)+1}=n \Leftrightarrow \frac{a'(a'+1)}{2(a'+1)}=n \Leftrightarrow a'=2n$$

which gives $b'=2n+1$ and $c'=1$. Hence a solution for a given $n\in\mathbb{N}$ is the infintie set of triples:

$$(a,b,c)=(2nd,2nd+d,d),\ d\in\mathbb{N}.$$

Also solved by DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; CÁSSIO DOS SANTOS SOUSA, Instituto Tecnológico de Aeronáutica, São Paulo, Brazil; KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA; and the proposers.

THE CONTEST CORNER

No. 6

Shawn Godin

The Contest Corner *est une nouvelle rubrique offerte pas* ***Crux Mathematicorum****, comblant ainsi le vide suite à la mutation en 2013 de* Mathematical Mayhem *et* Skoliad *vers une nouvelle revue en ligne. Il s'agira d'un amalgame de* Skoliad, The Olympiad Corner *et l'ancien* Academy Corner *d'il y a plusieurs années. Les problèmes en vedette seront tirés de concours destinés aux écoles secondaires et au premier cycle universitaire ; les lecteurs seront invités à soumettre leurs solutions ; ces solutions commenceront à paraître au prochain numéro.*

Les solutions peuvent être envoyées à : Shawn Godin, Cairine Wilson S.S., 975 Orleans Blvd., Orleans, ON, CANADA, K1C 2Z5 *ou par couriel à* `crux-contest@cms.math.ca`.

Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **1 decembre 2013**.

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5, 7 et 9, l'anglais précédera le français, et dans les numéros 2, 4, 6, 8 et 10, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Rolland Gaudet, de Université de Saint-Boniface, Winnipeg, MB, d'avoir traduit les problèmes.

CC26. Une fonction f vérifie $f(1) = 2$, et aussi, pour tout entier positif $n > 1$,

$$f(1) + f(2) + f(3) + \cdots + f(n) = n^2 f(n).$$

Déterminer la valeur de $f(2013)$.

CC27. Sur chacune des faces d'un cube $n \times n \times n$, on dessine un grillage de n^2 petits carrés. On trace alors un chemin de $(0, 0, 0)$ à (n, n, n) en utilisant, sans reculs, des côtés des petits carrés. Déterminer le nombre de tels chemins.

CC28. Le polynôme quartique $P(x)$ vérifie $P(1) = 0$. Aussi, il atteint sa valeur maximale de 3 au deux valeurs $x = 2$ et $x = 3$. Calculer $P(5)$.

CC29. Considérer trois parallélogrammes P_1, P_2 et P_3. Le parallélogramme P_3 se situe à l'intérieur du parallélogramme P_2, et ses sommets se trouvent sur les côtés de P_2. De façon similaire, le parallélogramme P_2 se situe à l'intérieur du parallélogramme P_1, et ses sommets se trouvent sur les côtés de P_1. Enfin, les côtés de P_3 sont parallèles aux côtés de P_1. Démontrer qu'un des côtés de P_3 a une longueur au moins la moitié de la longueur du côté parallèle de P_1.

CC30. Deux enfants égoïstes s'amusent au jeu suivant. Ils commencent avec un bol contenant N bonbons, le nombre N étant connu des deux enfants. Tour à tour, chaque enfant prend (si possible) un bonbon ou plus, mais sujet à la restriction de ne jamais prendre plus que la moitié des bonbons restants. Le gagnant n'est pas l'enfant ayant le plus grand nombre de bonbons à la fin, mais le dernier à pouvoir en prendre.
À titre d'exemple, s'il y a 3 bonbons au départ, le premier joueur doit en prendre un seul, car deux dépasse la moitié des bonbons disponibles à ce moment ; ensuite, le deuxième joueur est forcé d'en prendre un seul, laissant un bonbon dans le bol ; le premier joueur ne peut plus en prendre et perd le jeu.
(a) Démontrer que si le jeu démarre avec 2000 bonbons, alors le premier joueur gagne.
(b) Démontrer que si le jeu démarre avec $999\cdots999$ (2000 neufs) bonbons, alors le premier joueur gagne.

. . . .

CC26. A function f is defined in such a way that $f(1) = 2$, and for each positive integer $n > 1$,

$$f(1) + f(2) + f(3) + \cdots + f(n) = n^2 f(n).$$

Determine the value of $f(2013)$.

CC27. A $n \times n \times n$ cube has its faces ruled into n^2 unit squares. A path is to be traced on the surface of the cube starting at $(0, 0, 0)$ and ending at (n, n, n) moving only in a positive sense along the ruled lines. Determine the number of distinct paths.

CC28. The quartic polynomial $P(x)$ satisfies $P(1) = 0$ and attains its maximum value of 3 at both $x = 2$ and $x = 3$. Compute $P(5)$.

CC29. Consider three parallelograms P_1, P_2, P_3. Parallelogram P_3 is inside parallelogram P_2, and the vertices of P_3 are on the edges of P_2. Parallelogram P_2 is inside parallelogram P_1, and the vertices of P_2 are on the edges of P_1. The sides of P_3 are parallel to the sides of P_1. Prove that one side of P_3 has length at least half the length of the parallel side of P_1.

CC30. Two polite but vindictive children play a game as follows. They start with a bowl containing N candies, the number known to both contestants. In turn, each child takes (if possible) one or more candies, subject to the rule that no child may take, on any one turn, more than half of what is left. The winner is not the child who gets most candy, but the last child who gets to take some.
Thus, if there are 3 candies, the first player may only take one, as two would be more than half. The second player may take one of the remaining candies; and the first player cannot move and loses.
(a) Show that if the game begins with 2000 candies the first player wins.
(b) Show that if the game begins with $999\cdots999$ (2000 9's) candies, the first player wins.

THE OLYMPIAD CORNER

No. 304

Nicolae Strungaru

Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **1 decembre 2013**.

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5, 7 et 9, l'anglais précédera le français, et dans les numéros 2, 4, 6, 8 et 10, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal, d'avoir traduit les problèmes.

OC86. Lors d'une réunion contenant un nombre fini de participants, certains se trouvaient être des amis. Parmi tout groupe de 4 personnes, ou bien il y en avait 3 tous amis entre eux ou alors 3 tous inconnus l'un de l'autre. Montrer qu'on peut alors séparer tous les participants en deux groupes de sorte que le premier groupe ne contienne que des amis entre eux et que le second groupe ne contienne que des inconnus l'un de l'autre. (L'amitié est une relation mutuelle).

OC87. Appelons un nombre naturel n fidèle, s'il existe des nombres naturels $a < b < c$ tels que $a \mid b, b \mid c$ et $n = a + b + c$.

(i) Montrer que tous les nombres naturels sont fidèles sauf un nombre fini d'entre eux.

(ii) Trouver la somme de tous les nombres naturels qui ne sont pas fidèles.

OC88. Trouver toutes les fonctions $f : \mathbb{R} \mapsto \mathbb{R}$ telles que

$$f(f(x) - f(y)) = f(f(x)) - 2x^2 f(y) + f(y^2),$$

pour tous les $x, y \in \mathbb{R}$.

OC89. Soit $ABCD$ un quadrilatère cyclique qui ne soit pas un trapézoïde et dont les diagonales se coupent en E. Soit F et G les points milieux de AB et CD, et soit ℓ la droite par G parallèle à AB. Soit H et K les pieds des perpendiculaires issues de E sur les droites ℓ et CD. Montrer que $EF \perp HK$.

OC90. Soit n un entier positif. Si une racine de l'équation quadratique $x^2 - ax + 2n = 0$ est égale à

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}},$$

montrer que $2\sqrt{2n} \leq a \leq 3\sqrt{n}$.

. . . .

OC86. There were finitely many persons at a party among whom some were friends. Among any 4 of them there were either 3 who were all friends among each other or 3 who weren't friends with each other. Prove that you can separate all the people at the party in two groups in such a way that in the first group everyone is friends with each other and that all the people in the second group are not friends to anyone else in second group. (Friendship is a mutual relation).

OC87. Call a natural number n faithful if there exist natural numbers $a < b < c$ such that $a \mid b, b \mid c$ and $n = a + b + c$.

(i) Show that all but a finite number of natural numbers are faithful.

(ii) Find the sum of all natural numbers which are not faithful.

OC88. Find all functions $f : \mathbb{R} \mapsto \mathbb{R}$ such that

$$f(f(x) - f(y)) = f(f(x)) - 2x^2 f(y) + f(y^2),$$

for all $x, y \in \mathbb{R}$.

OC89. Let $ABCD$ be a cyclic quadrilateral which is not a trapezoid and whose diagonals meet at E. Let F and G be the midpoints of AB and CD, and let ℓ be the line through G parallel to AB. Let H, K be the feet of the perpendiculars from E onto the lines ℓ and CD. Prove that $EF \perp HK$.

OC90. Let n be a positive integer. If one root of the quadratic equation $x^2 - ax + 2n = 0$ is equal to

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}},$$

prove that $2\sqrt{2n} \leq a \leq 3\sqrt{n}$.

OLYMPIAD SOLUTIONS

OC26. Find all functions f from the real numbers to the real numbers which satisfy

$$f\left(x^3\right) + f\left(y^3\right) = (x+y)(f\left(x^2\right) + f\left(y^2\right) - f(xy))$$

for all real numbers x and y.
(*Originally question #3 from the 2008/9 British Mathematical Olympiad, Round 2.*)

Solved by Michel Bataille, Rouen, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Oliver Geupel, Brühl, NRW, Germany. We give the similar solutions of Bataille and Geupel.

For each real number a, the function $f(x) = ax$ is a solution. We prove there are no other solutions.

Let z be a non-negative real number. Setting $x = -y = \sqrt[3]{z}$ we get that $f(-x) = -f(x)$, thus f is an odd function. In particular $f(0) = 0$. Setting $y = 0$ in the given equation we get

$$f(x^3) = xf(x^2) \tag{1}$$

Substituting this in the functional equation we get

$$xf(x^2) + yf(y^2) = (x+y)(f(x^2) + f(y^2) - f(xy))$$

and thus

$$(x+y)f(xy) = xf(y^2) + yf(x^2) \tag{2}$$

Setting $y = 1$ and then $y = -1$ in (2) we get

$$(x+1)f(x) = xf(1) + f(x^2) \tag{3}$$

$$(x-1)f(-x) = xf(1) - f(x^2) \tag{4}$$

Adding (3) and (4) and using the fact that f is odd, we get

$$f(x) = xf(1)$$

which proves our claim.

OC27. A natural number k is said to be *n-squared* if, for every colouring of the squares in a chessboard of size $2n \times k$ with n colours, there are 4 squares with the same colour whose centres are the vertices of a rectangle with sides parallel to the sides of the chessboard.

For any given n, find the smallest natural number k which is n-squared.
(*Originally question #6 from the XXV Olimpiadi Italiane della Matematica.*)

Solved by Oliver Geupel, Brühl, NRW, Germany.

We show that the smallest n-squared number is

$$\binom{2n}{2} + 1\,.$$

First we give a colouring for a $2n \times \binom{2n}{2}$ chessboard that does not have 4 squares with the same colour whose centres are the vertices of a rectangle with sides parallel to the sides of the chessboard.

Let $P_1P_2\ldots P_{2n-1}$ be a regular $(2n-1)$–gon and consider the indices of vertices of the polygon modulo $2n-1$. Also let P_{2n} be an additional point (this one not to be modified modulo $2n-1$). For each $i = 1,\ 2,\ \ldots,\ 2n-1$, consider the set A_i of unordered pairs of vertices that have a common edge or diagonal that is parallel to P_iP_{i+1}. Moreover, let the pair $\{P_{n+i},\ P_{2n}\}$ be a member of A_i (where the index $n+i$ is taken modulo $2n-1$). Identifying pairs $\{P_r,\ P_s\}$ of edges with pairs $\{r,\ s\}$ of numbers from the set $\{1,\ 2,\ \ldots,\ 2n\}$, we obtain $2n-1$ disjoint sets $A_1,\ A_2,\ \ldots,\ A_{2n-1}$ of pairs of elements from $\{1,\ 2,\ \ldots,\ 2n\}$.

For each i, colour the elements of each pair in A_i with one of n colours $C_1,\ C_2,\ \ldots,\ C_n$, using the same colour for the elements of each pair and distinct colours for elements of distinct pairs. Apply the colouring of A_i to the i-th column of the chessboard. In this way we obtain a colouring of the leftmost $2n \times (2n-1)$ block B_1 of the board. It has the property that, if two cells in a column are equally coloured, then the similar cells in other columns are not equally coloured.

Consider the other $n-1$ blocks $B_2,\ B_3,\ldots,\ B_n$ of the board. A colouring of B_j is obtained from the colouring of B_1 by the permutation

$$\begin{pmatrix} C_1 & C_2 & \ldots & C_n \\ C_j & C_{j+1} & \ldots & C_{j+n-1} \end{pmatrix}$$

of the colours, where indices are taken modulo n. It has the property that, if two cells in a column are equaly coloured, then the similar cells in other columns are either not equally coloured or equally coloured but with a different colour.

We have therefore obtained a colouring for a $2n \times \binom{2n}{2}$ chessboard that does not have 4 squares with the same colour whose centres are the vertices of a rectangle with sides parallel to the sides of the chessboard.

It remains to prove that the number $\binom{2n}{2} + 1$ is n-squared.

Suppose that the squares of a $2n \times \left(\binom{2n}{2} + 1\right)$ chessboard are coloured with n colours. In each column we can choose at most n cells of different colours. The other at least n cells of the column can be combined with one of these cells to make an unordered pair of cells of the same colour. Therefore, we have at least n pairs of cells of the same colour in each column. Thus, we have at least $n \cdot \left(\binom{2n}{2} + 1\right)$ pairs of cells that are of the same colour and in the same column.

Since n colours are used, there is a colour C such that we have at least $\binom{2n}{2}+1$ pairs of cells whose members are in the same column and coloured with the colour C. But there are only $\binom{2n}{2}$ unordered pairs of rows. By the Pigeonhole principle, two of these pairs of cells must be in the same pair of rows. Therefore there must be 4 squares with the same colour whose centres are the vertices of a rectangle with sides parallel to the sides of the chessboard.

Consequently, $\binom{2n}{2}+1$ is n-squared.

OC28. A flea is initially at the point $(0,0)$ of the Euclidean plane. It then takes n jumps. Each jump is taken in any of the four cardinal directions (north, east, south or west). The first jump has length 1, the second jump has length 2, the third jump has length 4, and so on, the n^{th} jump has length 2^{n-1}.

Prove that if we know the number of jumps and the final position, we can uniquely determine the path the flea took.
(*Originally question #4 from XXV Olimpiadi Italiane della Matematica.*)

Solved by Oliver Geupel, Brühl, NRW, Germany.

Let $\mathbf{e_1}=(1,0)$, $\mathbf{e_2}=(0,1)$. Consider a vector $\mathbf{u}$ such that

$$\mathbf{u}=\sum_{k=0}^{n-1}2^k\mathbf{v_k}=\sum_{k=0}^{n-1}2^k\mathbf{w_k},\qquad \mathbf{v_k},\ \mathbf{w_k}\in\{\pm\mathbf{e_1},\ \pm\mathbf{e_2}\}.$$

We have to prove that

$$\mathbf{v_k}=\mathbf{w_k},\qquad k=0,1,\ldots,n-1. \tag{1}$$

The proof is by induction on n.

The claim (1) is clearly true if $n=1$. We now prove the inductive step.

If $\mathbf{v_{n+1}}=\mathbf{w_{n+1}}$, then

$$\sum_{k=1}^{n}2^k\mathbf{v_k}=\sum_{k=1}^{n}2^k\mathbf{w_k},$$

and the conclusion $\mathbf{v_k}=\mathbf{w_k}$, $k=0,1,\ldots,n$ follows by induction.

It remains to consider the case $\mathbf{v_{n+1}}\neq\mathbf{w_{n+1}}$. For $(x,y)\in\mathbb{R}^2$, let denote $|(x,y)|_1=|x|+|y|$. Then, it is easy to check that $|\mathbf{v_{n+1}}-\mathbf{w_{n+1}}|_1=2$. Thus, we obtain

$$\begin{aligned}2^{n+2}&=|2^{n+1}(\mathbf{v_{n+1}}-\mathbf{w_{n+1}})|_1=\left|\sum_{k=0}^{n}2^k(\mathbf{v_k}-\mathbf{w_k})\right|_1\le\sum_{k=0}^{n}|2^k(\mathbf{v_k}-\mathbf{w_k})|_1\\&\le\sum_{k=0}^{n}2^k\cdot 2=2^{n+2}-2,\end{aligned}$$

a contradiction. Hence, the case $\mathbf{v_{n+1}} \neq \mathbf{w_{n+1}}$ is impossible. This completes the induction.

OC29. Let $n \geq 3$ be a given integer, and $a_1, a_2, \cdots, a_n$ be real numbers satisfying $\min\limits_{1\leq i<j\leq n} |a_i - a_j| = 1$. Find the minimum value of $\sum\limits_{k=1}^{n} |a_k|^3$.
(*Originally question #4 from the 2009 Chinese Mathematical Olympiad.*)

Solved by Michel Bataille, Rouen, France; and Oliver Geupel, Brühl, NRW, Germany. We give the solution of Bataille.

Throughout this problem we will denote by

$$S(N) := \sum_{k=1}^{N} k^3 \,.$$

Let $A := \sum_{k=1}^{n} |a_k|^3$.

We show that the required minimum value is

$$A_{min} = \frac{1}{8} \left\lfloor \frac{n^2}{2} \right\rfloor \cdot \left\lfloor \frac{n^2-1}{2} \right\rfloor \,.$$

Since for $n = 2k$ we get $A = A_{min}$ for

$$(a_1,\ a_2,\ \ldots,\ a_n) = (-\frac{2m-1}{2}, \ldots, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \ldots, \frac{2m-1}{2})$$

and for $n = 2k+1$ we get $A = A_{min}$ for

$$(a_1,\ a_2,\ \ldots,\ a_n) = (-m,\ -(m-1), \ldots,\ -1, 0, 1, \ldots,\ m),$$

it suffices to show that $A \geq A_{min}$ for all $(a_1,\ a_2,\ \ldots,\ a_n)$ satisfying

$$\min_{1\leq i<j\leq n} |a_i - a_j| = 1 \,.$$

Since the condition is invariant under the permutations on a_j and under replacing all a_j by $-a_j$, without loss of generality we can assume that the number r of negative entries is not greater that the number of positive entries and that

$$a_1 \leq a_2 \leq a_2 \leq \cdots \leq a_r < 0 \leq a_{r+1} \leq \cdots \leq a_n \,.$$

If $r = 0$ then $a_1 \geq 0$ and $a_k \geq a_1 + (k-1)$ thus

$$A \geq a_1^3 + (a_1+1)^3 + \cdots + (a_1+n-1)^3 \geq S(n-1) \,.$$

Otherwise we have $1 \leq r \leq \frac{n}{2}$ and

$$a_1 < a_2 < \cdots < a_r < 0 \leq a_{r+1} < a_{r+2} < \cdots < a_n \,.$$

Then

$$a_{r+2} \geq a_{r+1} + 1,\ a_{r+3} \geq a_{r+1} + 2, \ldots, a_n \geq a_{r+1} + n - r - 1 \,,$$

and

$$a_{r-1} \leq a_r - 1,\ a_{r-2} \leq a_r - 2, \ldots, a_1 \leq a_r - r + 1\,.$$

Then we distinguish two cases:
If $a_{r+1} \geq 1$ then

$$\begin{aligned} A &\geq (r-1)^3 + (r-2)^3 + ... + 1^3 + 0^3 + 1^3 + ... + (n-r)^3 \\ &= S(r-1) + S(n-r) \end{aligned}$$

If $a_{r+1} < 1$ then since $-a_r > 1 - a_{r-1}$, we have

$$\begin{aligned} A \geq (r - a_{r+1})^3 + (r - 1 - a_{r+1})^3 + \cdots + (1 - a_{r+1})^3 + a_{r+1}^3 + (a_{r+1} + 1)^3 \\ + \cdots + (a_{r+1} + n - r - 1)^3 \end{aligned}$$

Now

$$(j - a_{r+1})^3 + (j + a_{r+1})^3 = 2j^3 + 6ja_{r+1}^2 \geq 2j^3\,,$$

for all $1 \leq j \leq r$. Thus, since $r \leq \lfloor \frac{n}{2} \rfloor$ we have

$$A \geq 2\sum_{j=1}^{r} j^3 + a_{r+1}^3 + (a_{r+1}+1)^3 + \cdots + (a_{r+1}+n-r-1)^3 \geq S(r) + S(n-r-1)$$

Thus, since $S(r-1) + S(n-r) - S(r) - S(n-r-1) = (n-r)^3 - r^3 \geq 0$, in both cases we have

$$A \geq S(r) + S(n-r-1)\,.$$

Now we look separately at the case n odd and n even.
Case 1: $n = 2m+1$ odd. Then $r \leq m$.
If $r = m$ then $S(r) + S(n-r-1) = 2S(m)$ and if $r \leq m-1$ then

$$\begin{aligned} S(r) + S(n-r-1) - 2S(m) &= S(2m-r) - S(m) - [S(m) - S(r)] \\ &= \sum_{k=1}^{m-r} [(m+k)^3 - (r+k)^3] \geq 0\,. \end{aligned}$$

Thus

$$A \geq 2S(m) = A_{min}\,.$$

Case 2: $n = 2m$ even. Then $r \leq m$.
If $r \leq m-1$ then $A \geq S(r) + S(2m-r-1)$.
If $r = m$, then we have

$$\begin{aligned} A \geq (m - a_{m+1})^3 + (m - 1 - a_{m+1})^3 + \cdots + (1 - a_{m+1})^3 + a_{m+1}^3 \\ + (a_{m+1}+1)^3 + \cdots + (a_{m+1} + m - 1)^3\,. \end{aligned}$$

Since

$$(m - a_{m+1})^3 + (a_{m+1} + m - 1)^3 \geq \frac{1}{4}(2m-1)^3$$
$$(m - 1 - a_{m+1})^3 + (a_{m+1} + m - 2)^3 \geq \frac{1}{4}(2m-1)^3$$
$$(m - 2 - a_{m+1})^3 + (a_{m+1} + m - 3)^3 \geq \frac{1}{4}(2m-1)^3$$
$$\vdots$$
$$(1 - a_{m+1})^3 + a_{m+1}^3 \geq \frac{1}{4}(1)^3$$

thus

$$A \geq \frac{1}{4}(1^3 + 3^3 + \cdots + (2m-1)^3) = A_{min}\,.$$

We complete the proof by observing that if $0 \leq r \leq m-1$ then

$$S(r) + S(2m - r - 1) \geq \frac{1}{4}(1^3 + 3^3 + \cdots + (2m-1)^3)\,. \tag{1}$$

Indeed, if $r = 0$ the inequality (1) is obvious, while for $r \geq 1$, $S(r) + S(2m - r - 1)$ can be rewritten as

$$[(2m-r-1)^3 + r^3] + [(2m-r-2)^3 + (r-1)^3] + \ldots + [(2m-2r)^3 + 1^3] + S(2m-2r-1)$$

and hence

$$\begin{aligned} &S(r) + S(2m - r - 1) \\ &\quad \geq \frac{1}{4}[(2m-1)^3(2m-3)^3 + \cdots + (2m-(2r-1))^3] + S(2m-2r-1) \\ &\quad \geq \frac{1}{4}[(2m-1)^3 + (2m-3)^3 + \cdots + 1^3]\,. \end{aligned}$$

Which completes the proof.

OC30. Let P be an interior point of a regular n-gon $A_1A_2\cdots A_n$. The lines A_iP meet $A_1A_2\cdots A_n$ at another point B_i, where $i = 1, 2, \cdots, n$. Prove that

$$\sum_{i=1}^{n} PA_i \geq \sum_{i=1}^{n} PB_i\,.$$

(*Originally question #8 from the 2008 China Western Mathematical Olympiad.*)

No solution to this problem was received.

BOOK REVIEWS

Amar Sodhi

A Wealth of Numbers: An Anthology of 500 Years of Popular Mathematics Writing. Edited by Benjamin Wardhaugh
Princeton University Press, 2012
ISBN: 978-0-691-14775-8, 370 +xv pp. US$45.00
Reviewed by ***S. Swaminathan****, Dalhousie University, Halifax, N. S.*

Almost everyone to whom you are introduced as a professor of mathematics is likely to respond "Oh! I was never good at math." Yet many such persons, the educated ones in particular, would love to read about mathematics, especially popular mathematics. During the second half of the last century Martin Gardner's articles on Mathematical Games and Recreations in *Scientific American* have inspired thousands of readers to delve more deeply into the large world of mathematics that he loved to explore and explain. Benjamin Wardhaugh's book reveals that popular mathematics writing didn't begin with Martin Gardner. In fact, it has a rich tradition stretching back hundreds of years. The book is an entertaining and enlightening anthology—the first of its kind—which presents nearly one hundred fascinating selections from the past 500 years of popular math writing, bringing to life a little-known side of math history. Ranging from the late fifteenth to the late twentieth century, and drawing from books, newspapers, magazines, and websites, the text includes recreational, classroom, and work mathematics; mathematical histories and biographies; accounts of higher mathematics; explanations of mathematical instruments; discussions of how math should be taught and learned; reflections on the place of math in the world; and math in fiction and humor.

There are eleven chapters, organized according to various themes which look at how mathematics was learnt and taught, used at work and played with in spare time, reflected on, and laughed about. Some chapters (1, 3, 5 and 7) are about mathematics done for fun: games, popularizations and histories. Others (2, 4, 6, and 8) show it in the class room and at work. Chapters 9 and 10 are more reflective, asking how mathematics should be taught, and why. The final chapter treats mathematics in fiction.

Featuring many tricks, games, problems, and puzzles, as well as much history and trivia, the selections include a sixteenth-century guide to making a horizontal sundial; "Newton for the Ladies" (1739); Leonhard Euler on the idea of velocity (1760); "Mathematical Toys" (1785); a poetic version of the rule of three (1792); "Lotteries and Mountebanks" (1801); Lewis Carroll on the game of logic (1887); "Maps and Mazes" (1892); "Einstein's Real Achievement" (1921); "Riddles in Mathematics" (1945); "New Math for Parents" (1966); and "PC Astronomy" (1997). Each selection is placed in context by a brief introduction.

Thus this anthology is a unique window into the hidden history of popular mathematics, The book will provide many hours of fun and learning to anyone who loves popular mathematics and science.

The Irrationals: a Story of the Numbers You Can't Count On by Julian Havil
Princeton University Press, Princeton & Oxford, 2012
ISBN: 978-0-691-14342-2, Cloth, 298 + ix pages, US$29.95
Reviewed by ***Edward Barbeau****, University of Toronto, Toronto, ON*

What are the real numbers? This is a deep question. It has taken about two millenia to understand and explain their properties, and there is still much to learn. The convoluted history of the real number system is the subject of this interesting book by Julian Havil, his fourth exposition published by Princeton. It is a pleasure to read, with solid mathematical content balanced by historical notes, philosophical discussion and anecdote.

The story begins with the Greeks whose idea of number was tied to counting, so that they had a separate concept of magnitude to deal with the continuum. This dichotomy was forced upon them by the discovery of incommensurable geometric magnitudes whose ratio could not be expressed with whole numbers. As Havil says, "the first significant implication of what we now call irrationality was that mathematical enquiry became geometric enquiry, an approach that was to pervade all European mathematics and last well into the eighteenth century." (p. 35-36) But there were other perspectives. After a generous discussion of Greek mathematics that includes Euclid's treatment of ratio, proportionality and incommensurability, Havil reviews the contributions of Indian, Levantine and early Renaissance European mathematicians. These investigators were comfortable with a number system that embraced surds, which they manipulated stunningly. While the spirit of these developments is preserved, the mathematics is presented in modern notation.

The next three chapters of the book treat particular irrationals, in particular e, π and $\zeta(3)$. Determining the character of these numbers occupied the attention of many leading mathematicians and involved a variety of tools: continued fractions, integration and infinite series. A full chapter is devoted to the proof of the irrationality of $\zeta(3)$ by the French mathematician, Roger Apéry in 1978. This is a real *tour de force* and the author presents the right amount of detail to convey the essence of the argument without bogging the reader down in technicalities.

The author moves to more general and theoretical considerations, especially the degree of closeness of rational approximations to real numbers. Here among familiar material, I found some that was new and fascinating. The basic question is when a real number α can be approximated by infinitely many rationals p/q with $|\alpha - p/q| < 1/cq^r$ where c and r are given reals exceeding 1. The whole story is far from being wrapped up in the early result that α is irrational if and only if there are infinitely many rationals with $|\alpha - p/q| < 1/q^2$. Havil surveys the theoretical evolution of degree of rational approximation up to the striking result of Klaus Roth in 1955 that, given an algebraic number α and positive ϵ, there are only finitely many rationals p/q for which $|\alpha - p/q| < 1/q^{2+\epsilon}$.

There are many other topics touched upon: straightedge-and-compasses construction, the Gelfond-Schneider theorem, uncountability of the reals (by a nested interval rather than a diagonal argument), approximation using continued fractions, the Lagrange and Markov spectra, "randomness" of decimal digits, rigorous formulation of the real number system. The final chapter, entitled "Does Irrationality Matter?" contains brief musings on the well-tempered musical scale, the golden ratio, Penrose tiles, the characteristic function

$$\lim_{m\to\infty}\lim_{n\to\infty}\cos^{2n}(m!\pi x)$$

of the rationals, dynamical systems and Samuel Beatty's notable result that, for positive irrationals x and y whose reciprocals add to 1, the sequences $\{\lfloor mx \rfloor\}$ and $\{\lfloor ny \rfloor\}$ together contain each positive integer exactly once. (It is slightly annoying to see Beatty described as an American mathematician, despite being described as being from the University of Toronto.)

On the whole, the author has presented a coherent and compelling account of a sophisticated topic to which he has evidently devoted a lot of time and effort to research and sort out. However, there were several places where I found the treatment heavy-handed. There is an easier way of establishing the incommensurability of side and diagonal of a square than is presented on page 24. On page 79, a modulo 4 argument suffices to show that $a^2 + b^2 = 3c^2$ is not solvable in integers. The argument on page 146 that the least common multiple of the first n natural numbers does not exceed n raised to the number of primes not greater than n is unnecessarily complicated. More seriously, on page 86 appears the assertion that, for a positive continuous function defined on the nonnegative reals and with absolute maximum M_N on $[0, N]$,

$$\int_0^1 f(x)dx = \lim_{N\to\infty}\frac{\sum_{r=0}^{N} f(r)}{M_N(N+1)}$$

(try this on $f(x) = \sin(\pi x) + 2$); fortunately, this is applied to a situation where it works.

Despite some minor lapses and rough edges, this is a book worth having. It is accessible to secondary students keen on mathematics and anyone with a first-year university background, although it requires sustained concentration in parts. I particularly recommend it for undergraduate students and secondary teachers.

PROBLEM SOLVER'S TOOLKIT

No. 1

Shawn Godin

The Problem Solver's Toolkit is a new feature in ***Crux Mathematicorum****. It will contain short articles on topics of interest to problem solvers at all levels. Occasionally, these pieces will span several issues.*

Fermat's Little Theorem

A number of results from elementary number theory are useful for solving contest-type problems. A standard theorem is

Fermat's Little Theorem *If p is a prime and a is a positive integer, then*

$$a^p \equiv a \pmod{p}.$$

If a is an integer not divisible by p, then

$$a^{p-1} \equiv 1 \pmod{p}.$$

We can see this in action for $p = 7$ below, where all numbers are calculated modulo 7. Notice that both versions are illuminated by the last two columns.

a	a^2	a^3	a^4	a^5	a^6	a^7
0	0	0	0	0	0	0
1	1	1	1	1	1	1
2	4	1	2	4	1	2
3	2	6	4	5	1	3
4	2	1	4	2	1	4
5	4	6	2	3	1	5
6	1	6	1	6	1	6

Now let's use this theorem in a problem. In the seventh season episode *Treehouse of Horror VI* of *The Simpsons* (first aired October 30 1995), during the segment *Homer 3D*, dim witted protagonist Homer Simpson is transported into three dimensional space. While there, he encounters a number of mathematical objects, like geometric solids and the coordinate axes, as well as a number of equations that float by. One of the equations states

$$1782^{12} + 1841^{12} = 1922^{12},$$

which violates Fermat's Last Theorem (which had just been proved two years prior to this episode airing after being unsolved for more than 350 years!). The example is large enough that it will probably fool your calculator. My calculator, with a 10 digit display, has $1782^{12}+1841^{12}$ and 1922^{12} agreeing to all displayed digits as does $1922^{12}-1841^{12}$ and 1782^{12}. Warning lights might have gone on if I calculated $1922^{12}-1782^{12}$ and 1841^{12} and noticed that they differed in the last decimal place. Yet, my calculator gives the following results:

$$\sqrt[12]{1782^{12}+1841^{12}} = 1922,$$
$$\sqrt[12]{1922^{12}-1841^{12}} = 1782,$$
$$\sqrt[12]{1922^{12}-1782^{12}} = 1841.$$

Although computer algebra software like Maple and Mathematica would make short work of this problem, since none of 1782, 1841 and 1922 is a multiple of 13, we can apply Fermat's theorem to note that

$$1782^{12}+1841^{12} \equiv 1+1 \not\equiv 1 \equiv 1922^{2} \pmod{13}.$$

Thus the equation is false.

Many proofs of Fermat's Little Theorem exist; my favourite uses properties of modular arithmetic. Consider the set $\{1,2,3,\ldots,p-1\}$, and let a be a positive integer with $1 \le a < p$; then if we reduce the set $\{a,2a,3a,\ldots,(p-1)a\}$ modulo p, it is just a permutation of the original set. This is true since no element of the new set is divisible by p since all factors in the products are less than p. Similarly, there are no duplications in the set, since if $am \equiv an \pmod{p}$, then $m \equiv n \pmod{p}$. Thus

$$\begin{aligned}(a)(2a)(3a)\cdots((p-1)a) &\equiv (1)(2)(3)\cdots(p-1) \pmod{p}\\ \Rightarrow a^{p-1}(1)(2)(3)\cdots(p-1) &\equiv (1)(2)(3)\cdots(p-1) \pmod{p}\\ \Rightarrow a^{p-1} &\equiv 1 \pmod{p}\end{aligned}$$

It is important to note that the converse of Fermat's little theorem is false. That is, if $a^{n-1} \equiv 1 \pmod{n}$ it doesn't mean that n is prime. A number n that satisfies $a^{n-1} \equiv 1 \pmod{n}$, for some positive integer a, yet is not prime, is called a *pseudoprime in base a*. An example would be $n = 341 = 11 \times 31$ which is a pseudoprime base 2. There even exist extremal pseudoprimes, that is, numbers n that are pseudoprime to all bases a that are relatively prime to them. That is, n satisfies $a^{n-1} \equiv 1 \pmod{n}$ for every positive integer a, $1 < a < n$, such that $\gcd(a,n) = 1$. These numbers are called *Carmichael numbers*, the smallest of which is $561 = 3 \times 11 \times 17$.

Try your hand with the following problems.

Problems:

1. Can you spot a much easier proof that $1782^{12}+1841^{12} \neq 1922^{12}$? After finding it, feel free to slap yourself in the head and say "D'oh!". You can check out other examples of mathematics in *The Simpsons* at [2].

2. In the season 10 episode *The Wizard of Evergreen Terrace* of *The Simpsons* that first aired September 20, 1998, Homer is seen writing on a chalkboard which contains the equation $3987^{12} + 4365^{12} = 4472^{12}$ (along with a demonstration of how to transform a torus into a sphere). Show that, once again, this equation is incorrect. You can check out other examples of "Fermat near misses" at [1].

3. Show that 341 is a pseudoprime base 2.

4. Pick several values of a, with $1 < a < 561$ and $\gcd(561, a) = 1$ and show that 561 is a pseudoprime base a. You can check out pseudoprimes, Carmichael numbers and everything prime at [3].

5. Prove that $n^{20} - 1$ is divisible by 11 for all positive integers relatively prime to 11.

6. Show that 129 is **not** prime using Fermat's little theorem. **Hint**: Evaluate $2^{128} \pmod{129}$.

References

[1] Noam D. Elkies, *Tables of Fermat "near-misses" approximate solutions of* $x^n + y^n = z^n$ *in integers with* $0 < x <= y < z < 223$ *and* n *in* $[4, 20]$, `http://www.math.harvard.edu/ elkies/ferm.html`.

[2] Andrew Nestler, *Guide to Mathematics and Mathematicians on The Simpsons*, `http://homepage.smc.edu/nestler_andrew/SimpsonsMath.htm`.

[3] Paulo Ribenboim, *The New Book of Prime Number Records*, third edition, Springer-Verlag, New York, 1996.

RECURRING CRUX CONFIGURATIONS 7

J. Chris Fisher

Triangles whose angles satisfy $B = 2C$

As usual, we denote the lengths of the sides opposite angles A, B, and C of triangle ABC by a, b, and c, respectively. The main result for this month has appeared repeatedly on the pages of **Crux** since the beginning:

Theorem. Angle B is twice angle C in ΔABC if and only if the sides satisfy $b^2 = c(a + c)$.

Five proofs can be found in [1976: 73-74], and further proofs in [1984: 287], [2001: 416], [2002: 400-401], and [2006: 159]. My favorite was the fifth of the 1976 proofs, devised by the editor Léo Sauvé. Curiously the same proof appeared the same year in [3], and reappeared in these pages in 2002. Let's bring it back again now. Extend AB beyond B to D so that $BD = a$, and consider the triangles ABC and ACD. Because they share $\angle A$, they are similar exactly when $\frac{b}{c} = \frac{a+c}{b}$, which is equivalent to $\angle ACD = \angle B$. But $\angle B$ is an exterior angle of ΔDCB, whence $\angle B = 2\angle BCD$ and $\angle ACD = \angle C + \frac{1}{2}\angle B$. Putting it all together we get $b^2 = c(a + c)$ if and only if $\angle B = \angle C + \frac{1}{2}\angle B$, as required.

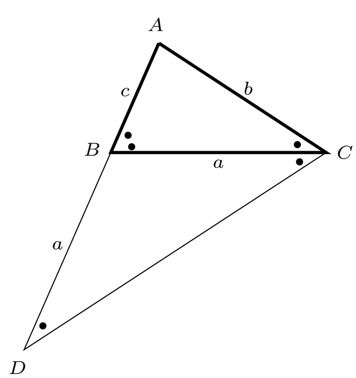


Figure 1: In ΔABC, $\angle B = 2\angle C$ if and only if $b^2 = c(a + c)$.

Integer-sided triangles with $\angle \mathbf{B} = \mathbf{2}\angle \mathbf{C}$. A list of all integer-sided triangles with $\angle B = 2\angle C$ is obtained using parameters k, m, and n, where k is an arbitrary positive integer while m and n are relatively prime integers satisfying $2n > m > n > 1$; then

$$a = k(m^2 - n^2), \quad b = kmn, \quad c = kn^2.$$

The smallest three examples are therefore

n	m	a	b	c
2	3	5	6	4
3	4	7	12	9
3	5	16	15	9

Proofs of the claim can be found in [2006: 159] and [2009: 335]. The former is the solution to an Olympiad problem (XXI Albanian Mathematical Olympiad 2000, 12th Form, third round, Problem 5 [2004: 344]), which demanded further that the length of a be a prime. For that to happen we must have $k = 1$, $m = n + 1$, and n chosen so that $a = 2n + 1$ is a prime number. Virtually the same proofs are to be found in [1] (which supplies further references to this theorem and to the generalization to triangles with $\angle B = k\angle C$) and in [3]. The earliest reference supplied by readers was [2], which called for the smallest integer-sided triangle for which one angle is twice another; there is little doubt, however, that the investigation of 4-5-6 triangles goes back long before 1960. Richard Guy assumed that readers were familiar with the properties of integer triangles with one angle twice another in his Problem 1220 [1987: 54; 1988: 125-126], a problem designed to amuse puzzle enthusiasts long before Sudoku became popular. It consisted of an 8 by 8 square partitioned into 64 blank squares; about half of these squares were assigned numbers from 1 to 37. The 64 squares were to be filled with digits; the twenty clues that accompanied the diagram were triples (a, b, c) of two- and three-digit numbers that form sides of primitive integer triangles ABC with angle B twice the size of angle A.

I found eight further **Crux** problems dealing with triangles having one angle twice another.

Problem 5 of the Second Stage Exam of the 10th Iranian Mathematical Olympiad [1995: 9; 1996: 265-267]. In triangle ABC with $A \leq 90°$ and $B = 2C$, let the bisector of $\angle C$ intersect the median AM (M is the midpoint of BC) at D; then $\angle MDC \leq 45°$, with equality if and only if $A = 90°$.

Problem 2302 [1998: 45; 1999: 53-55] (proposed by Toshio Seimiya). Determine angles B and C of ΔABC if for the foot D of the internal bisector of $\angle A$ we have $AB + AD = CD$ and $AC + AD = BC$.

The answer is that $C = 20°$ and $B = 40°$. The first featured solution used a picture to prove directly that $\angle B = 2\angle C$; other solvers first showed that $b^2 = c(a + c)$.

Problem 2501 [2000: 45; 2001: 52-53] (proposed by Toshio Seimiya). In ΔABC the internal bisectors of angles A and B meet the opposite sides at D and E respectively; then $AB + BD = AE + EB$ if and only if $\angle B = 2\angle C$ or $\angle B = 120°$.

Problem 2559 [2000: 305; 2001: 466-467] (proposed by Hojoo Lee). Triangle ABC has incentre I; $AB + BI = AC$ if and only if $\angle B = 2\angle C$.

Problem 2568 [2000: 373; 2001: 416; 2002: 400-401] (proposed by K. R. S. Sastry). If the sides of ΔABC satisfy $b^2 = ac + c^2$ and $a^2 = cb + b^2$, find the measures of its angles.

From our main theorem we easily deduce that the angles are $C = \frac{\pi}{7}, B = \frac{2\pi}{7}$, and $A = \frac{4\pi}{7}$.

Problem 2936 [2004: 174; 2005: 189-190] (proposed by Toshio Seimiya). If in ΔABC we have $\angle B = 2\angle C$ and $\angle A > 90°$, and D is the point where the perpendicular to AC through C meets AB, then

$$\frac{1}{AB} - \frac{1}{BD} = \frac{2}{BC}.$$

Problem 2942 [2004: 229, 232; 2005: 246-247] (proposed by Toshio Seimiya). If in ΔABC we have $\angle B = 2\angle C$ and D is the point on the ray CB such that $\angle ADC = \frac{1}{2}\angle BAC$, then

$$\frac{1}{AC} + \frac{1}{CD} = \frac{1}{AB}.$$

Problem 3377 [2008: 430, 432; 2009: 468-469] (proposed by Toshio Seimiya). Let ABC be a triangle with $\angle B = 2\angle C$, and let D be the foot of the interior bisector of $\angle A$ while M and N are the midpoints of AC and BD, respectively. If A, M, D, and N are concyclic, then $\angle A = 72°$.

References

[1] D. Kipp Johnson, Solution to Problem 578 (proposed by Robert Patenaude, May 1996), *College Mathematics Journal*, **28**:3 (May 1997) 233-235.

[2] D. O. Shklarsky, N. N. Khentzov, and I. M. Yaglom, *The USSR Olympiad Problem Book*, Dover, New York, 1962. Problem 129(a).

[3] William Wynne Willson, **60.9** A Generalisation of a Property of the 4,5,6 Triangle, *Mathematical Gazette* **60**: 412 (Jun 1976) 130-131.

PROBLEMS

Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **1 decembre 2013***. Une étoile* ($\star$) *après le numéro indique que le problème a été soumis sans solution.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5, 7, et 9, l'anglais précédera le français, et dans les numéros 2, 4, 6, 8, et 10, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Rolland Gaudet, de Université de Saint-Boniface, Winnipeg, MB, d'avoir traduit les problèmes.

3731. *Proposé par Richard K. Guy, Université de Calgary, Calgary, AB.*

Les longueurs des côtés d'un quadrilatre sont $AB = 5$, $BC = 10$, $CD = 11$ et $DA = 14$.

(a) Si le quadrilatère est cyclique, quel est le diamètre de son cercle circonscrit ?

(b) Si on modifie l'ordre des sommets, ceci affecte-t-il la réponse en (a) ?

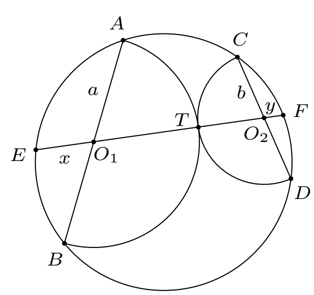
3752. *Proposé par Péter Ivády, Budapest, Hongrie.*

Démontrer que si $n \geq 2$ est un entier positif alors l'inégalité suivante tient

$$\frac{1}{2}\left[1+\frac{1}{n}\left(1-\frac{1}{n}\right)\right]^2 < \left(1-\frac{1}{2^3}\right)\left(1-\frac{1}{3^3}\right)\cdots\left(1-\frac{1}{n^3}\right) .$$

3753. *Proposé par Abdilkadir Altintaş, mathematics teacher, Emirdağ, Turkey.*

Des demi cercles avec centres O_1 et O_2 sont tracés à partir des cordes AB et CD d'un certain cercle Γ, ces demi cercles étant tangents au point T. La ligne passant par O_1 et O_2 intersecte Γ aux points E et F. Si $O_1A = a$, $O_2C = b$, $O_1E = x$ et $O_2F = y$, démontrer que $a - b = x - y$.



3754. *Proposé par Šefket Arslanagić, Université de Sarajevo, Sarajevo, Bosnie et Herzégovine.*

Soit un triangle scalène ΔABC. Dénotons par w_a, w_b et w_c les longueurs des bissectrices d'angles de ΔABC, par R le rayon de son cercle circonscrit et par r le rayon de son cercle inscrit. Démontrer l'inégalité suivante

$$576\sqrt{3}r^3 < \frac{w_a^2 - w_b^2}{b-a} + \frac{w_b^2 - w_c^2}{c-b} + \frac{w_c^2 - w_a^2}{a-c} < 72\sqrt{3}R^3$$

3755. *Proposé par Bill Sands, Université de Calgary, Calgary, AB.*

Déterminer tous les nombres réels $a \leq b \leq c \leq d$ en progression arithmétique et satisfaisant aux deux équations $a+b+c+d=1$ et $a^2+b^2+c^2+d^2=d$.

3756. *Proposé par Michel Bataille, Rouen, France.*

Soit le triangle ABC inscrit dans un cercle Γ et soit M le mipoint de son arc BC qui ne contient pas A. Les perpendiculaires à AB passant par M et à MB passant par B intersectent au point K ; aussi, les perpendiculaires à AC passant par M et à MC passant par C intersectent au point L. Démontrer que les lignes BC et AM intersectent au mipoint de KL.

3757. *Proposé par Dragoljub Milošević, Gornji Milanovac, Serbie.*

Dénotons par A, B et C les angles d'un triangle, mesurés en radians, par s son demi périmètre, par R le rayon de son cercle circonscrit et par r le rayon de son cercle inscrit. Démontrer les inégalités suivantes.

(a) $\displaystyle \frac{1}{A} + \frac{1}{B} + \frac{1}{C} \leq \frac{9}{2\pi} \cdot \frac{R}{r}$;

(b) $\displaystyle \left(\frac{A}{B} + \frac{B}{C} + \frac{C}{A}\right)^3 \leq \frac{9}{2\pi} \cdot \frac{R}{r}$.

3758. *Proposé par Paul Yiu, Florida Atlantic University, Boca Raton, FL, É-U.*

À partir d'un point X sur un segment BC, construire un point A tel que le cercle inscrit du triangle ABC touche BC au point X et tel que la ligne reliant le point Gergonne et les points Nagel du triangle est parallèle à BC.

3759. *Proposé par Nguyen Minh Ha, Hanoi, Vietnam.*

Pour un polygone convexe $A_1A_2\cdots A_n$ et un point intérieur P, soit $a_i = \sum_{j=1}^{n} A_iA_j$. Démontrer que $\sum_{i=1}^{n} PA_i < \max_{1\leq j\leq n}\{a_j\}$.

3760. *Proposé par Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Soit $p \geq 2$ un entier. Déterminer la limite

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{\sqrt[p]{n}}{\sum_{j=1}^{p} \sqrt[p]{k^j (n+k)^{p-j+1}}}.$$

. . . .

3751. *Proposed by Richard K. Guy, University of Calgary, Calgary, AB.*

The edge lengths of a quadrilateral are $AB = 5$, $BC = 10$, $CD = 11$, $DA = 14$.

(a) If the quadrilateral is cyclic, what is the diameter of its circumcircle?

(b) If we alter the order of the edges, does it affect the answer to (a)?

3752. *Proposed by Péter Ivády, Budapest, Hungary.*

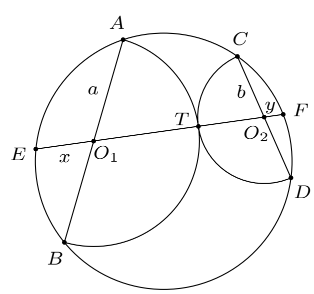
Show that if $n \geq 2$ is a positive integer then

$$\frac{1}{2}\left[1 + \frac{1}{n}\left(1 - \frac{1}{n}\right)\right]^2 < \left(1 - \frac{1}{2^3}\right)\left(1 - \frac{1}{3^3}\right)\cdots\left(1 - \frac{1}{n^3}\right)$$

holds.

3753. *Proposed by Abdilkadir Altintaş, mathematics teacher, Emirdağ, Turkey.*

Semi-circles with centres O_1 and O_2 are drawn on chords AB and CD of a circle Γ such that they are tangent at T. The line through O_1 and O_2 intersects Γ at E and F. If $O_1A = a$, $O_2C = b$, $O_1E = x$ and $O_2F = y$, show that $a - b = x - y$.



3754. *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Prove that in all scalene triangles ΔABC the inequality

$$576\sqrt{3}r^3 < \frac{w_a^2 - w_b^2}{b - a} + \frac{w_b^2 - w_c^2}{c - b} + \frac{w_c^2 - w_a^2}{a - c} < 72\sqrt{3}R^3$$

holds, where w_a, w_b and w_c are the lengths of the angle bisectors; R is the radius of the circumcircle; and r is the inradius of ΔABC.

3755. *Proposed by Bill Sands, University of Calgary, Calgary, AB.*

Find all real numbers $a \leq b \leq c \leq d$ which form an arithmetic progression which satisfy the two equations $a + b + c + d = 1$ and $a^2 + b^2 + c^2 + d^2 = d$.

3756. *Proposed by Michel Bataille, Rouen, France.*

Let triangle ABC be inscribed in circle Γ and let M be the midpoint of the arc BC of Γ not containing A. The perpendiculars to AB through M and to MB through B intersect at K and the perpendiculars to AC through M and to MC through C intersect at L. Prove that the lines BC, AM intersect at the midpoint of KL.

3757★. *Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia.*

Let A, B, C be the angles (measured in radians), s the semi-perimeter, R the circumradius and r the inradius of a triangle. Prove that

(a) $\dfrac{1}{A} + \dfrac{1}{B} + \dfrac{1}{C} \leq \dfrac{9}{2\pi} \cdot \dfrac{R}{r}$;

(b) $\left(\dfrac{A}{B} + \dfrac{B}{C} + \dfrac{C}{A}\right)^3 \leq \dfrac{9}{2\pi} \cdot \dfrac{R}{r}$.

3758. *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Given a point X on the segment BC, construct a point A such that the incircle of triangle ABC touches BC at X, and that the line joining the Gergonne point and the Nagel poits of the triangle is parallel to BC.

3759. *Proposed by Nguyen Minh Ha, Hanoi, Vietnam.*

Given a convex polygon $A_1A_2\cdots A_n$ with an interior point P. Let $a_i = \displaystyle\sum_{j=1}^{n} A_iA_j$. Prove that $\displaystyle\sum_{i=1}^{n} PA_i < \max_{1\leq j\leq n}\{a_j\}$.

3760. *Proposed by Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Let $p \geq 2$ be an integer. Determine the limit

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{\sqrt[p]{n}}{\sum_{j=1}^{p} \sqrt[p]{k^j (n+k)^{p-j+1}}}.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3651. [2011 : 318, 320] *Correction. Proposed by Hung Pham Kim, student, Stanford University, Palo Alto, CA, USA.*

Let a, b, and c be nonnegative real numbers such that $a+b+c = 3$. Prove that

$$a^2b + b^2c + c^2a + abc + 4abc(3 - ab - bc - ca) \le 5\,.$$

Comment by Stan Wagon, Macalester College, St. Paul, MN, USA.

Using *Mathematica*, the correct maximum for the function is not 4, but the negative of the first root of

$$87\,979\,149+709\,313\,125x+502\,089\,984x^2+138\,280\,800x^3+17\,105\,664x^4+800\,000x^5$$

which is about 4.936. The actual values of a, b, c that realize this are combinations of roots of degree 15 polynomials.

It was also observed by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, Missouri, USA that the problem, as originally stated, was incorrect. Arslanagić found a counterexample showing the original problem was incorrect and asks for which $k > 0$ does the inequality $a^2b+b^2c+c^2a+abc+kabc(3-ab-bc-ca) \le 4$ hold?

3652. [2011 : 318, 321] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let α and β be positive real numbers. Find the value of

$$\lim_{n\to\infty}\prod_{k=1}^{n}\left(1+\frac{k^\alpha}{n^\beta}\right)\,.$$

Solution by the proposer, modified by the editor.

Let $\alpha + 1 > \beta$. Since

$$\begin{aligned}\prod_{k=1}^{n}\left(1+\frac{k^\alpha}{n^\beta}\right) &> 1+\sum_{k=1}^{n}\frac{k^\alpha}{n^\beta} > 1+\frac{1}{n^\beta}\int_0^n x^\alpha dx\\ &= 1+\frac{n^{\alpha+1-\beta}}{\alpha+1},\end{aligned}$$

it follows that
$$\lim_{n\to\infty}\prod_{k=1}^{n}\left(1+\frac{k^\alpha}{n^\beta}\right)=+\infty.$$

Now suppose that $\alpha+1\le\beta$ and let
$$x_n=\log\prod_{k=1}^{n}\left(1+\frac{k^\alpha}{n^\beta}\right).$$

Then
$$x_n=\sum_{k=1}^{n}\log\left(1+\frac{k^\alpha}{n^\beta}\right)=\sum_{k=1}^{n}\left[\frac{\log(1+k^\alpha/n^\beta)}{k^\alpha/n^\beta}\right]\left[\frac{k^\alpha}{n^\beta}\right].$$

Define a_n to be the maximum and b_n to be the minimum for $1\le k\le n$ of $(k^\alpha/n^\beta)^{-1}\log(1+k^\alpha/n^\beta)$. Observe that $t^{-1}\log(1+t)$ is monotone and tends to 1 as t decreases to 0, and that $\lim_{n\to\infty}n^\alpha/n^\beta=0$. Therefore $\lim_{n\to\infty}a_n=\lim_{n\to\infty}b_n=1$. It follows from
$$a_n\sum_{k=1}^{n}\frac{k^\alpha}{n^\beta}\le x_n\le b_n\sum_{k=1}^{n}\frac{k^\alpha}{n^\beta}$$

that
$$\lim_{n\to\infty}x_n=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^\alpha}{n^\beta}.$$

When $\beta=\alpha+1$,
$$\begin{aligned}\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^\alpha}{n^\beta}&=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^\alpha\\&=\int_0^1 x^\alpha dx=\frac{1}{\alpha+1}.\end{aligned}$$

When $\beta>\alpha+1$, then
$$\begin{aligned}\sum_{k=1}^{n}\frac{k^\alpha}{n^\beta}&<\frac{1}{n^\beta}\int_0^{n+1}x^\alpha dx=\frac{(n+1)^{\alpha+1}}{n^\beta(\alpha+1)}\\&\le\frac{(2n)^{\alpha+1}}{n^\beta(\alpha+1)}=\frac{2^{\alpha+1}}{(\alpha+1)n^{\beta-\alpha-1}}.\end{aligned}$$

It follows that $\lim_{n\to\infty}x_n=0$.

Therefore
$$\lim_{n\to\infty}\prod_{k=1}^{n}\left(1+\frac{k^\alpha}{n^\beta}\right)=\begin{cases}+\infty, & \text{if } \alpha+1>\beta,\\ e^{1/(\alpha+1)}, & \text{if } \alpha+1=\beta,\\ 0, & \text{if } \alpha+1<\beta.\end{cases}$$

The correct answer was obtained by ANASTASIOS KOTRONIS, Athens, Greece; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, MO, USA; and the proposer. Unfortunately, the first three solutions were flawed. There was one incorrect solution.

3653. [2011 : 318, 321] *Proposed by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

Let O be the centre of a sphere S circumscribing a tetrahedron $ABCD$. Prove that:

(i) there exists tetrahedra whose four faces are obtuse triangles; and

(ii) ⋆ if O is inside or on $ABCD$, then at least two faces of $ABCD$ are acute triangles.

Solution to (i) by the proposer.

Consider the isosceles trapezoid $ABCD$ with $AD||BC$ and obtuse angles $\angle BAC$ and $\angle BDC$. Note that all four triangles BCD, CDA, DAB, ABC are obtuse. Now lift B and D a short distance above the plane so that the lines AC and BD are skew; then $ABCD$ becomes a tetrahedron with no face that is an acute triangle.

(ii) *Remains open.*

We received no satisfactory solution to part (ii), so we have moved this problem to our unsolved problems file. In fact, there seems to be something wrong with its statement: If A and B are the north and south poles of a sphere, and points C and D are chosen near A on the sphere so that $\angle DAC$ is obtuse, then the faces ABC and ABD are right triangles (with right angles at C and D), and ACD is obtuse. Yet the circumcentre lies on the edge AB of the tetrahedron, contrary to the second claim. Perhaps one needs to have the circumcentre strictly inside the tetrahedron to insure that at least two of its faces be acute.

3654. [2011 : 319, 321] *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Let a, b, c, and d be nonnegative real numbers such that $a^2+b^2+c^2+d^2=1$. Prove that

$$a^3+b^3+c^3+d^3+abc+bcd+cda+dab \;\le\; 1\,.$$

Solution by Albert Stadler, Herrliberg, Switzerland.

Let $f(a,b,c,d)=a^3+b^3+c^3+d^3+abc+bcd+cda+dab$. By the Cauchy-Schwarz Inequality,

$$a^2+bc\le\sqrt{a^2+b^2}\sqrt{a^2+c^2}. \tag{1}$$

So, by equation (1), followed by a second application of the Cauchy-Schwarz inequality, and by twice using the given condition $\sum\limits_{\text{cyclic}} a^2=a^2+b^2+c^2+d^2=1$, the desired result is obtained as follows:

$$\begin{aligned}f(a,b,c,d) &= \sum_{\text{cyclic}} a(a^2+bc) \le \sum_{\text{cyclic}} a\sqrt{a^2+b^2}\sqrt{a^2+c^2}\\ &\le \sqrt{\sum_{\text{cyclic}} a^2}\sqrt{\sum_{\text{cyclic}}(a^2+b^2)(a^2+c^2)}\\ &= 1\cdot\sqrt{\sum_{\text{cyclic}}(a^4+a^2b^2+b^2c^2+c^2a^2)} = a^2+b^2+c^2+d^2 = 1\end{aligned}$$

Additionally, for a, b, c, and d nonnegative and $a^2+b^2+c^2+d^2=1$, observe that the function $f(a,b,c,d)$ attains its maximum value of 1, for example, at the point $(a,b,c,d)=(1,0,0,0)$.

Also solved by AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; STAN WAGON, Macalester College, St. Paul, MN, USA; HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, Missouri, USA; and the proposer.

Wang and Wojdylo used the Lagrange Multiplier Method and Gröbner bases to prove the result and showed that $(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)$, *and* $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$ *were the only places where a maximum was attained. Wagon pointed out that the problem can be completely solved using standard algorithms in* Mathematica

3655. [2011 : 319, 321] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Calculate the integral

$$\int_0^1\int_0^1 x\left\{\frac{1}{1-xy}\right\}dxdy,$$

where $\{a\}=a-\lfloor a\rfloor$ denotes the fractional part of a.

I. Solution by Roger Zarnowski, Angelo State University, San Angelo, TX, USA.

The value is $1-\pi^2/12$. To see this, let

$$(u,v)=\left(\frac{1}{1-xy},\frac{y}{x}\right).$$

For $0<x,y<1$, we have that $u>1$ and $v>0$. The inverse transformation is given by

$$(x,y)=\left(\sqrt{\frac{u-1}{uv}},\sqrt{\frac{(u-1)v}{u}}\right).$$

The Jacobian of the transformation is

$$\begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{vmatrix} = \begin{vmatrix}\dfrac{1}{2u^2v}\sqrt{\dfrac{uv}{u-1}} & -\dfrac{u-1}{2uv^2}\sqrt{\dfrac{uv}{u-1}}\\[2ex] \dfrac{v}{2u^2}\sqrt{\dfrac{u}{(u-1)v}} & \dfrac{u-1}{2u}\sqrt{\dfrac{u}{(u-1)v}}\end{vmatrix} = \frac{1}{2u^2v}.$$

The unit square $S = \{(x,y) : 0 < x, y < 1\}$ is covered by hyperbolic arcs of equation $xy = \alpha$ where $0 < \alpha < 1$. These arcs are mapped into vertical segments of equation $u(1-\alpha) = 1$. The square S is mapped onto the region

$$R = \left\{(u,v) : \frac{u-1}{u} < v < \frac{u}{u-1}, 1 < u < \infty\right\}.$$

Then

$$\begin{aligned}\int_0^1\int_0^1 x\left\{\frac{1}{1-xy}\right\}dxdy &= \int_1^\infty \int_{\frac{u-1}{u}}^{\frac{u}{u-1}} \sqrt{\frac{u-1}{uv}}\{u\}\frac{1}{2u^2v}dvdu \\ &= \sum_{n=1}^\infty \int_n^{n+1}\int_{\frac{u-1}{u}}^{\frac{u}{u-1}} \sqrt{\frac{u-1}{u}}\{u\}\frac{1}{2u^2}v^{-3/2}dvdu.\end{aligned}$$

For $n \le u < n+1$, we have $\{u\} = u - n$, so that upon integrating with respect to v, we obtain

$$\begin{aligned}-\sum_{n=1}^\infty \int_n^{n+1} \sqrt{\frac{u-1}{u}}(u-n)\frac{1}{u^2}\left[v^{-1/2}\right]_{\frac{u-1}{u}}^{\frac{u}{u-1}} du & \\ &= -\sum_{n=1}^\infty \int_n^{n+1} \sqrt{\frac{u-1}{u}}(u-n)\frac{1}{u^2}\left(\sqrt{\frac{u-1}{u}} - \sqrt{\frac{u}{u-1}}\right)du \\ &= -\sum_{n=1}^\infty \int_n^{n+1} (u-n)\frac{1}{u^2}\left(\frac{u-1}{u} - 1\right)du \\ &= \sum_{n=1}^\infty \int_n^{n+1} (u^{-2} - nu^{-3})du = \sum_{n=1}^\infty \frac{1}{2n(n+1)^2} \\ &= \frac{1}{2}\sum_{n=1}^\infty \left[\frac{1}{n} - \frac{1}{n+1} - \frac{1}{(n+1)^2}\right] \\ &= \frac{1}{2}\left[\sum_{n=1}^\infty\left(\frac{1}{n} - \frac{1}{n+1}\right) - \left(\sum_{n=1}^\infty \frac{1}{n^2} - 1\right)\right] \\ &= \frac{1}{2}\left[1 - \left(\frac{\pi^2}{6} - 1\right)\right] = 1 - \frac{\pi^2}{12}.\end{aligned}$$

II. Solution by Missouri State University Problem Solving Group, Springfield, MO, USA, expanded by the editor.

Note that, if $\frac{n-1}{n} \le xy < \frac{n}{n+1}$, then $\lfloor \frac{1}{1-xy} \rfloor = n$. Thus the desired integral is equal to

$$\sum_{n=1}^\infty \left(\int_{\frac{n-1}{n}}^1 \int_{\frac{n-1}{nx}}^1 x\left(\frac{1}{1-xy} - n\right)dydx - \int_{\frac{n}{n+1}}^1 \int_{\frac{n}{(n+1)x}}^1 x\left(\frac{1}{1-xy} - n\right)dydx\right),$$

or

$$\int_0^1 \int_0^1 \left(\frac{x}{1-xy}\right) dydx - n\left(\int_{\frac{n}{n+1}}^1 \int_{\frac{n}{(n+1)x}}^1 xdydx - \int_{\frac{n-1}{n}}^1 \int_{\frac{n-1}{nx}}^1 xdydx\right).$$

We have that

$$\begin{aligned}\int_0^1 \int_0^1 \left(\frac{x}{1-xy}\right) dydx &= \lim_{t\to 0^+} \int_0^{1-t} \int_0^1 \left(\frac{x}{1-xy}\right) dydx \\ &= \lim_{t\to 0^+} \int_0^{1-t} \left[-\ln(1-xy)\right]_0^1 dx \\ &= \lim_{t\to 0^+} \int_0^{1-t} [-\ln(1-x)]dx = \lim_{t\to 0^+} \left[(1-x)\ln(1-x)+x\right]_0^{1-t} \\ &= \lim_{t\to 0^+} [t\ln t + (1-t)] = 1,\end{aligned}$$

and, for each $q \in [0,1]$,

$$\int_q^1 \int_{q/x}^1 xdydx = \int_q^1 (x-q)dx = \frac{1}{2}(1-q)^2.$$

Therefore the desired integral is equal to

$$\begin{aligned}&1 - \frac{1}{2}\sum_{n=1}^{\infty} n\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) = 1 - \frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{1}{n} - \frac{1}{n+1} + \frac{1}{(n+1)^2}\right] \\ &= 1 - \frac{1}{2}\left[1 + \sum_{n=1}^{\infty}\frac{1}{(n+1)^2}\right] = 1 - \frac{\pi^2}{12}\end{aligned}$$

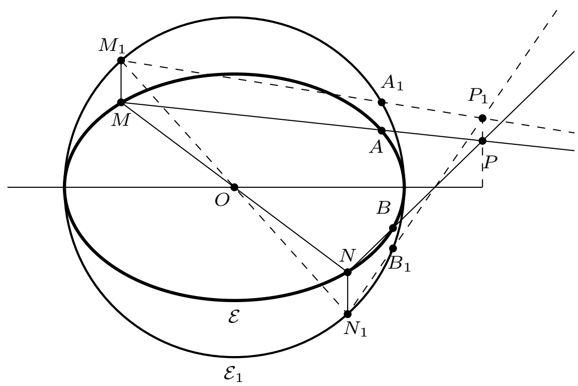
Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; PAUL DEIERMANN, Southeast Missouri State University, Cape Girardeau, MO, USA; ALBERT STADLER, Herrliberg, Switzerland; and the proposer. Deiermann partitioned the domain of integration into regions between hyperbolic arcs, while Curtis followed the strategy of the first solution using the transformation $(x,t) = (x,(1-xy)^{-1})$. There were two incorrect solutions, each of which yielded the putative answer $1-\gamma$, where γ is Euler's constant.

3656. [2011 : 319, 321] *Proposed by Michel Bataille, Rouen, France.*

Let AB be a fixed chord of an ellipse that is not a diameter and let MN be a variable diameter. Show that the locus of the intersection of MA and NB is an ellipse with the same eccentricity as that of the original ellipse, and find a geometrical description of its centre.

Solution by the proposer.

As usual, the line MA is the tangent to the given ellipse $\mathcal{E}$ at A if $M = A$ and, similarly, NB is the tangent at B if $N = B$. We denote by $\mathbf{S}$ the reflection in the centre O. Without loss of generality, we suppose that coordinates have been chosen so that $\mathcal{E}$ has equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $a > b > 0$. Let $\mathcal{E}_1$ denote the circle with centre O and radius a, and $\mathcal{A}$ be the strain (perspective affinity) that takes the point $K_1(x, y)$ to $K(x, \frac{by}{a})$ (so that $\mathcal{A}(\mathcal{E}_1) = \mathcal{E}$). Let A_1, B_1, M_1, N_1 be the points of $\mathcal{E}_1$ with respective images A, B, M, N under $\mathcal{A}$. From the properties of perspective affinities, $N_1 = \mathbf{S}(M_1)$; moreover, if A_1M_1 and B_1N_1 intersect at P_1, then AM and BN intersect at $P = \mathcal{A}(P_1)$, as in the accompanying figure.



We will soon prove that the locus of P_1 is a circle Γ_1 orthogonal to $\mathcal{E}_1$ through A_1 and B_1. It will then follow that the desired locus Γ is just $\mathcal{A}(\Gamma_1)$. For its eccentricity, let $U_1(\alpha, \beta)$ and r be the centre and radius of Γ_1, so that Γ_1 has equation $(x-\alpha)^2+(y-\beta)^2 = r^2$. Then the equation of Γ is $(x-\alpha)^2+(\frac{ay}{b}-\beta)^2 = r^2$, or

$$\frac{(x-\alpha)^2}{r^2} + \frac{(y - \frac{\beta b}{a})^2}{\frac{b^2r^2}{a^2}} = 1.$$

Thus, Γ is an ellipse centred at $U(\alpha, \frac{\beta b}{a}) = \mathcal{A}(U_1)$; its eccentricity is

$$\sqrt{1 - \frac{\frac{b^2r^2}{a^2}}{r^2}} = \sqrt{1 - \frac{b^2}{a^2}},$$

which equals the eccentricity of $\mathcal{E}$, as desired. Since $U = \mathcal{A}(U_1)$ the construction of U is routine once U_1 has been obtained. Since Γ_1 is orthogonal to $\mathcal{E}_1$ at A_1 and B_1, U_1 must be the point of intersection of the perpendicular bisector of A_1B_1 and the tangent to $\mathcal{E}_1$ at A_1. [*Editor's comment*: Alternatively, U can be constructed directly as the point of intersection of the tangents to $\mathcal{E}$ at A and B.]

It remains to prove the claim that the locus of $P_1 = A_1M_1 \cap B_1N_1$ is a circle through A_1 and B_1 that is orthogonal to the circle $\mathcal{E}_1$. Let M_1 be any point on $\mathcal{E}_1$. If $M_1 = B_1$ then $P_1 = B_1$; if $M_1 = \mathbf{S}(A_1)$, then $P_1 = A_1$. Otherwise, P_1 is not on A_1B_1, so the inversion with centre at P_1 that interchanges M_1 with A_1 takes N_1 to B_1 and fixes $\mathcal{E}_1$. Since M_1N_1 is a diameter of $\mathcal{E}_1$, the circle $P_1A_1B_1$, which is the image under the inversion of the line M_1N_1, is therefore orthogonal to $\mathcal{E}_1$, and we have $P_1 \in \Gamma_1$. Conversely, let X be an arbitrary point on Γ_1; we must show that there exists a point $M_1 \in \mathcal{E}_1$ with $N_1 = \mathbf{S}(M_1)$ such that $X = A_1M_1 \cap B_1N_1$. If $X = B_1$, then X is obtained from $M_1 = B_1$; if $X = A_1$, then X is obtained from $M_1 = \mathbf{S}(\mathcal{A}_1)$. Otherwise, we take M_1 as the second point of intersection of XA_1 and $\mathcal{E}_1$. With $N_1 = \mathbf{S}(M_1)$, the image of N_1 under the inversion with centre X that interchanges M_1 and A_1 is on $\mathcal{E}_1$ and XN_1, hence is B_1. Consequently, every point $X \in \Gamma_1$ is a point of the locus.

We received no other solutions. Readers with some knowledge of projective geometry will note that because the points M and N are projectively related, the line AM in the line pencil A is projectively related to the line BM in the line pencil B, whence the locus $AM \cap BN$ is a conic through A and B by Steiner's construction. (See, for example, H. S. M. Coxeter, Projective Geometry*, Theorem 8.51.) One easily sees that the locus is bounded, so that the conic must be an ellipse, but coordinates are needed to prove that the new ellipse has the same eccentricity as the original.*

3657. [2011 : 319, 321] *Proposed by Thanos Magkos, 3rd High School of Kozani, Kozani, Greece.*

Prove that for the angles of any triangle the following inequality holds

$$\frac{\cos^2 A}{1+\cos^2 A} + \frac{\cos^2 B}{1+\cos^2 B} + \frac{\cos^2 C}{1+\cos^2 C} \geq \frac{1}{2}.$$

I. Solution by Salem Malikić, student, Simon Fraser University, Burnaby, BC.

Let $a = \cos^2 A$, $b = \cos^2 B$, and $c = \cos^2 C$, then the inequality becomes

$$\frac{a}{1+a} + \frac{b}{1+b} + \frac{c}{1+c} \geq \frac{1}{2}$$

which is equivalent to

$$\frac{a+b+c+2(ab+bc+ca)+3abc}{(1+a)(1+b)(1+c)} \geq \frac{1}{2}$$

or

$$\frac{a+b+c+3(ab+bc+ca)+5abc-1}{(1+a)(1+b)(1+c)} \geq 0.$$

Thus, it suffices to prove that

$$a+b+c+3(ab+bc+ca)+5abc \geq 1. \tag{1}$$

Without loss of generality, assume that $A = \min\{A, B, C\}$. Then $A \leq \frac{\pi}{3}$ implying $\cos A \geq \frac{1}{2}$ so $\sqrt{a} \geq \frac{1}{2}$. Hence,

$$3(ab+ca) \geq 3 \cdot 2\sqrt{a^2bc} = 6\sqrt{a}\sqrt{abc} \geq 3\sqrt{abc} \geq 2\sqrt{abc}. \tag{2}$$

It is well known that

$$\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C = 1$$

that is,

$$a+b+c+2\sqrt{abc} = 1. \tag{3}$$

Using (2) and (3) we obtain

$$\begin{aligned} a+b+c+3(ab+bc+ca)+5abc &\geq a+b+c+3(ab+ca) \\ &\geq a+b+c+2\sqrt{abc}=1 \end{aligned}$$

establishing (1) and completing the proof.

II. Solution by Titu Zvonaru, Comăneşti, Romania.

Consider first the case when the triangle is right angled; say, $C=\frac{\pi}{2}$. Since $\cos B=\cos\left(\frac{\pi}{2}-A\right)=\sin A$ the given inequality is equivalent, in succession, to

$$\begin{aligned} \frac{\cos^2 A}{1+\cos^2 A}+\frac{\sin^2 A}{1+\sin^2 A} &\geq \frac{1}{2} \\ 2\cos^2 A+4\sin^2 A\cos^2 A+2\sin^2 A &\geq 1+\cos^2 A+\sin^2 A+\sin^2 A\cos^2 A \\ 3\sin^2 A\cos^2 A &\geq 0 \end{aligned}$$

which is clearly true.

Now suppose $A,B,C\neq\frac{\pi}{2}$. Applying the identity $\frac{1}{\cos^2\theta}=1+\tan^2\theta$ with $\theta=A,B$, and C, the given inequality becomes

$$\frac{1}{2+\tan^2 A}+\frac{1}{2+\tan^2 B}+\frac{1}{2+\tan^2 C}\geq\frac{1}{2}$$

or by setting $x=\tan A$, $y=\tan B$, and $z=\tan C$,

$$\frac{1}{2+x^2}+\frac{1}{2+y^2}+\frac{1}{2+z^2}\geq\frac{1}{2}\,. \tag{4}$$

Clearing the denominators and using the well known fact that $x+y+z=xyz$, (4) is equivalent, in succession, to

$$\begin{aligned} \sum_{\text{cyclic}}(8+4x^2+4y^2+2x^2y^2) &\geq 8+\sum_{\text{cyclic}}(4x^2+2x^2y^2)+x^2y^2z^2 \\ 16+4x^2+4y^2+4z^2 &\geq x^2y^2z^2 \\ 16+4(x^2+y^2+z^2) &\geq (x+y+z)^2 \\ 16+3(x^2+y^2+z^2)-2(xy+yz+zx) &\geq 0 \end{aligned}$$

and finally,

$$16+x^2+y^2+z^2+(x-y)^2+(y-z)^2+(z-x)^2\geq 0$$

which is clearly true.

Note that the given inequality is always strict unless we include degenerated triangles in which case equality would hold if and only if two of A, B, and C equal $\frac{\pi}{2}$ and the other one equals 0.

Also solved by AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MARIAN DINCĂ, *Bucharest, Romania; OLEH FAYNSHTEYN, Leipzig, Germany; EDMUND SWYLAN, Riga, Latvia; and HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, Missouri, USA.*

3658. [2011 : 319, 322] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $-\pi < \theta_0 < \theta_1 < \cdots < \theta_k < \pi$ and let a_j, $j = 0, 1, \cdots, k$, be complex numbers. Prove that if

$$\lim_{n\to\infty} \sum_{j=0}^{k} a_j \cos(\theta_j n) = 0\,,$$

then $a_j = 0$ for all j.

Solution by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy.

The conclusion is false as shown by the counterexample below:

Let $k = 1$, $\theta_0 = -\frac{\pi}{4}$, $\theta_1 = \frac{\pi}{4}$. Then

$$\lim_{n\to\infty} \sum_{j=0}^{k} a_j \cos(\theta_j n) = \lim_{n\to\infty} (a_0 + a_1) \cos\left(\frac{n\pi}{4}\right) = (a_0 + a_1) \lim_{n\to\infty} \cos\left(\frac{n\pi}{4}\right) = 0$$

which implies that $a_0 + a_1 = 0$, not necessarily $a_0 = a_1 = 0$.

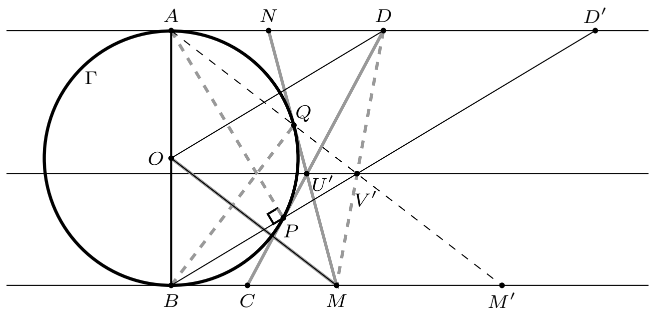
No other solutions were received. The proposer actually realized his error after the problem had already appeared.

3659. [2011 : 319, 322] *Proposed by Michel Bataille, Rouen, France.*

Let P be a point on a circle Γ with diameter AB. The tangent to Γ at P intersects the tangents at A and B in D and C, respectively. Let M be any point of the line BC and V the point of intersection of MD and BP. If the parallel to BC through V meets CD in U, show that the line MU is tangent to Γ.

Solution by Edmund Swylan, Riga, Latvia.

When $M = B$ or $M = C$ the result is trivially true (with MU touching Γ at B and at P, respectively). So, let us assume that $M \neq B, C$ and let the second tangent to Γ from M touch it at Q. When MQ is not parallel to CD, define $U' := MQ \cap CD$ and let V' be the point where the parallel to BC through U' meets BP. We must prove that $V = V'$ (since that implies $U' = U \in MQ$); since V' is on BP by definition, this amounts to proving that $V' \in MD$.



Denote the centre of Γ by O, and let $D' := AD \cap BP$ and $N := AD \cap MQ$. Because AB is a diameter we have $BP \perp AP$; because OD bisects the angle between the tangents from D to A and P, AP is also perpendicular to OD. Thus, $OD \parallel BP$ and, because O is the midpoint of AB, it follows that D is the midpoint of AD'. Similarly, define $M' := BC \cap AQ$ and, because $OM \parallel AQ$ we have M is the midpoint of BM'.

We have $U'Q = U'P$ because they are tangents to Γ from U'. Moreover, the triangles $U'PV'$ and CPB are homothetic, and because the tangents CB and CP are equal in length, we deduce that $U'V' = U'P$ and, therefore, $U'Q = U'V'$. This implies that $\Delta QU'V' \sim \Delta QNA$ by SAS ($\angle QU'V' = \angle QNA$ because Q lies on the transversal of the parallel sides $U'V'$ and NA, while $\frac{U'V'}{U'Q} = \frac{NA}{NQ} = 1$). Consequently, $V' \in AQ$, whence there is a dilatation centred at V' that takes $\Delta V'AD'$ and its median $V'D$ to $\Delta V'M'B$ and its median $V'M$. We conclude that $V' \in MD$, as desired.

Our argument falls apart when $MQ \parallel CD$, in which case U' is not defined. Still, the result continues to hold for this position of M since $CDNM$ forms a rhombus that is circumscribed about Γ, in which case the figure is symmetric about O and we have $MQ \parallel CD$ if and only if $MD \parallel BP$; we then interpret U' to be the common point of infinity of MQ and CD if and only if V' is the common point of infinity of MD and BP, which is precisely what was to be proved.

Also solved by TITU ZVONARU, Cománeşti, Romania; and the proposer.

None of our correspondents mentioned that the result is actually a theorem of projective geometry: The circle Γ can be replaced by any conic. Moreover, $A, B, P,$ and Q can be any four points on the conic — the centre O is irrelevant. The result reduces to Pascal's theorem applied to the quadrangle $ABPQ$ with two of its vertices repeated: If $A, B, P,$ and Q are four points of a conic then the tangents at A and B, the tangents at P and Q, the sides AP and BQ, and the sides BP and AQ meet in collinear points. For the special case of our problem, we infer that the line joining U (where the tangents at P and Q meet) to V ($= AQ \cap BP$) is parallel to the tangents at A and B and it passes through $W := AP \cap BQ$. The same theorem applied to the tangents at B and Q (which meet at M) and at A and P (which meet at D) says that V lies also on MD. As a bonus line MD also contains the point $AB \cap PQ$, and the line joining that point to C also contains W and N.

3660. [2011 : 319, 322] *Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia.*

Triangle ABC has inradius r, circumradius R, and side lengths a, b, c. Prove that

$$\frac{y+z}{x} \cdot \frac{1}{a^2} + \frac{z+x}{y} \cdot \frac{1}{b^2} + \frac{x+y}{z} \cdot \frac{1}{c^2} \geq \frac{1}{Rr}\,,$$

for all positive real numbers x, y and z.

Solution by Kee-Wai Lau, Hong Kong, China.

Let s be the semiperimeter of the triangle ABC. Using the well known facts that $abc = 4sRr$ and $p + q \geq 2\sqrt{pq}$ for any nonnegative real numbers p and q, we have

$$\begin{aligned}
&\frac{y+z}{x} \cdot \frac{1}{a^2} + \frac{z+x}{y} \cdot \frac{1}{b^2} + \frac{x+y}{z} \cdot \frac{1}{c^2} \\
&\qquad = \left(\frac{y}{x} \cdot \frac{1}{a^2} + \frac{x}{y} \cdot \frac{1}{b^2}\right) + \left(\frac{z}{y} \cdot \frac{1}{b^2} + \frac{y}{z} \cdot \frac{1}{c^2}\right) + \left(\frac{x}{z} \cdot \frac{1}{c^2} + \frac{z}{x} \cdot \frac{1}{a^2}\right) \\
&\qquad \geq \frac{2}{ab} + \frac{2}{bc} + \frac{2}{ca} = \frac{2(a+b+c)}{abc} = \frac{1}{Rr}.
\end{aligned}$$

Also solved by ARKADY ALT, San Jose, CA, USA; AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; PRITHWIJIT DE, Homi Bhabha Centre for Science Education, Mumbai, India; MARIAN DINCĂ, Bucharest, Romania; JOHN G. HEUVER, Grande Prairie, AB; DIMITRIOS KOUKAKIS, Kato Apostoloi, Greece; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; TITU ZVONARU, Comăneşti, Romania; and the proposer.