PROBLEMS FOR DECEMBER

Solutions should be submitted to
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no later than **January 31, 2001**.

49. Find all ordered pairs (x, y) that are solutions of the following system of two equations (where a is a parameter):

$$x - y = 2$$

$$\left(x - \frac{2}{a}\right)\left(y - \frac{2}{a}\right) = a^2 - 1 \ .$$

Find all values of the parameter a for which the solutions of the system are two pairs of nonnegative numbers. Find the minimum value of $x + y$ for these values of a.

50. Let n be a natural number exceeding 1, and let A_n be the set of *all* natural numbers that are not relatively prime with n (*i.e.*, $A_n = \{x \in \mathbf{N} : \ \gcd\,(x, n) \neq 1\}$. Let us call the number n *magic* if for each two numbers $x, y \in A_n$, their sum $x + y$ is also an element of A_n (*i.e.*, $x + y \in A_n$ for $x, y \in A_n$).

(a) Prove that 67 is a magic number.

(b) Prove that 2001 is **not** a magic number.

(c) Find all magic numbers.

51. In the triangle ABC, $AB = 15$, $BC = 13$ and $AC = 12$. Prove that, for this triangle, the angle bisector from A, the median from B and the altitude from C are concurrent (*i.e.*, meet in a common point).

52. One solution of the equation $2x^3 + ax^2 + bx + 8 = 0$ is $1 + \sqrt{3}$. Given that a and b are rational numbers, determine its other two solutions.

53. Prove that among any 17 natural numbers chosen from the sets $\{1, 2, 3, \cdots, 24, 25\}$, it is always possible to find two whose product is a perfect square.

54. A circle has exactly one common point with each of the sides of a $(2n + 1)-$sided polygon. None of the vertices of the polygon is a point of the circle. Prove that at least one of the sides is a tangent of the circle.