PROBLEMS FOR OCTOBER

Solutions should be submitted to
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no later than **November 30, 2000**.

37. Let ABC be a triangle with sides a, b, c, inradius r and circumradius R (using the conventional notation). Prove that

$$\frac{r}{2R} \leq \frac{abc}{\sqrt{2(a^2+b^2)(b^2+c^2)(c^2+a^2)}} \ .$$

When does equality hold?

38. Let us say that a set S of nonnegative real numbers if *hunky-dory* if and only if, for all x and y in S, either $x+y$ or $|x-y|$ is in S. For instance, if r is positive and n is a natural number, then $S(n,r) = \{0, r, 2r, \cdots, nr\}$ is hunky-dory. Show that every hunky-dory set with finitely many elements is $\{0\}$, is of the form $S(n,r)$ or has exactly four elements.

39. (a) $ABCDEF$ is a convex hexagon, each of whose diagonals AD, BE and CF pass through a common point. Must each of these diagonals bisect the area?

(b) $ABCDEF$ is a convex hexagon, each of whose diagonals AD, BE and CF bisects the area (so that half the area of the hexagon lies on either side of the diagonal). Must the three diagonals pass through a common point?

40. Determine all solutions in integer pairs (x, y) to the diophantine equation $x^2 = 1 + 4y^3(y+2)$.

41. Determine the least positive number p for which there exists a positive number q such that

$$\sqrt{1+x} + \sqrt{1-x} \leq 2 - \frac{x^p}{q}$$

for $0 \leq x \leq 1$. For this least value of p, what is the smallest value of q for which the inequality is satisfied for $0 \leq x \leq 1$?

42. G is a connected graph; that is, it consists of a number of vertices, some pairs of which are joined by edges, and, for any two vertices, one can travel from one to another along a chain of edges. We call two vertices *adjacent* if and only if they are endpoints of the same edge. Suppose there is associated with each vertex v a nonnegative integer $f(v)$ such that all of the following hold:
(1) If v and w are adjacent, then $|f(v) - f(w)| \leq 1$.
(2) If $f(v) > 0$, then v is adjacent to at least one vertex w such that $f(w) < f(v)$.
(3) There is exactly one vertex u such that $f(u) = 0$.
Prove that $f(v)$ is the number of edges in the chain with the fewest edges connecting u and v.