Solutions and Comments

7. Let
$$S = \frac{1^2}{1\cdot 3} + \frac{2^2}{3\cdot 5} + \frac{3^2}{5\cdot 7} + \cdots + \frac{500^2}{999\cdot 1001} \ .$$

Find the value of S.

Solution. [Miranda Holmes] Each term of S is of the form
$$\frac{i^2}{(2i-1)(2i+1)} = \frac{1}{4}\left(\frac{i}{2i-1} + \frac{i}{2i+1}\right) \ .$$

Thus
$$S = \frac{1}{4}\sum_{i=1}^{500}\left(\frac{i}{2i-1} + \frac{i}{2i+1}\right) \ .$$
$$4S = \sum_{i=1}^{500}\left(\frac{i}{2i-1} + \frac{i}{2i+1}\right) = \frac{1}{1} + \frac{1}{3} + \frac{2}{3} + \frac{2}{5} + \frac{3}{5} + \frac{3}{7} + \cdots + \frac{500}{999} + \frac{500}{1001} \ .$$

Since
$$\frac{i}{2i+1} + \frac{i+1}{2i+1} = 1 \ ,$$
we can combine all such terms to get
$$4S = 1 + 499 + \frac{500}{1001} = \frac{501000}{1001} \ .$$

Thus,
$$S = \frac{125250}{1001} \ .$$

8. The sequences $\{a_n\}$ and $\{b_n\}$ are such that, for every positive integer n,
$$a_n > 0 \ , \quad b_n > 0 \ , \quad a_{n+1} = a_n + \frac{1}{b_n} \ , \quad b_{n+1} = b_n + \frac{1}{a_n} \ .$$

Prove that $a_{50} + b_{50} > 20$.

Solution. Note that $x + (1/x) \geq 2$ for every positive real x. Consider the sequence $c_n = (a_n + b_n)^2$. Since
$$c_2 = (a_2 + b_2)^2 = \left(a_1 + \frac{1}{b_1} + b_1 + \frac{1}{a_1}\right)^2 = \left[\left(a_1 + \frac{1}{a_1}\right) + \left(b_1 + \frac{1}{b_1}\right)\right]^2 \ ,$$
we find that $c_2 \geq (2+2)^2 = 16$.

For each positive integer n,
$$\begin{aligned} c_{n+1} &= (a_{n+1} + b_{n+1})^2 = \left(a_n + \frac{1}{b_n} + b_n + \frac{1}{a_n}\right)^2 \\ &= [(a_n^2 + b_n^2 + 2a_nb_n) + \left(\frac{1}{a_n^2} + \frac{1}{b_n^2} + 2\cdot\frac{1}{a_n}\cdot\frac{1}{b_n}\right) \\ &\qquad + 2\cdot a_n\cdot\frac{1}{a_n} + 2\cdot b_n\cdot\frac{1}{b_n} + 2\left(\frac{a_n}{b_n} + \frac{b_n}{a_n}\right) \\ &= c_n + \left(\frac{1}{b_n} + \frac{1}{a_n}\right)^2 + 2 + 2 + 2\cdot\left(\frac{a_n}{b_n} + \frac{b_n}{a_n}\right) > c_n + 8 \ , \end{aligned}$$

Thus, $c_{n+1} > c_n + 8$. It follows that

$$c_{50} > c_{49} + 8 > c_{48} + 2 \cdot 8 > \cdots > c_2 + 48 \cdot 8 \geq 16 + 48 \cdot 8 = 400 \ .$$

Since $a_{50} + b_{50} > 0$, it follows that $a_{50} + b_{50} > 20$.

9. There are six points in the plane, no three of them collinear. Any three of them are vertices of a triangle whose sides are of different length. Prove that there exists a triangle whose smallest side is the largest side of another triangle.

Comment. Before giving the solution to the problem, we present a result that you should be aware of; pay also close attention to the proof.

Proposition. There are six points in the plane or in space, no three of them collinear. Each of the segments between two of them is coloured in one of two colours. There exists a triangle whose vertices are three of the given points and whose sides are of the same colour.

Proof. Let A be one of the points. There are five segments joining A to the other points. Since they have one of two colours, by the Pigeonhole Principle, at least three of them must have the same colour. Wolog, suppose that AB, AC, AD are coloured the same. If any of the segments BD, BC, CD has this colour, then we will have a triangle in this colour. Otherwise, BCD must be a triangle all of whose edges have the other colour.

Note 1: The minimum number of points with such a property is 6. If there are five points, it is possible to colour the segments between any two of them so that a triangle with edges of a single colour does not exist. For example, for a regular pentagon, we can colour all the sides with one colour and all the diagonals with the other.

Note 2: A triangle of one-colour always exists when we have 17 points in the plane (no three collinear) and three colours are used for the segments. This can be given a similar proof. ¿From any point, at least six of the segments emanating from it have the same colour. Now look at the six points terminating these segments.

Now we can solve the problem.

Solution. Consider the six points and all triangles whose vertices are any three of them. Colour the (uniquely determined) largest side of each triangle black, and colour the remaining edges red. There must be a triangle all of whose edges are the same colour. This colour cannot be red. (Why?) So there must be a triangle all of whose edges are black; its smallest edge must be the largest edge of some other triangle.

10. In a rectangle, whose sides are 20 and 25 units of length, are placed 120 squares of side 1 unit of length. Prove that a circle of diameter 1 unit can be placed in the rectangle, so that it has no common points with the squares.

Solution. [Miranda Holmes] If a circle of diameter 1 can be placed, it means that there must be a point in the rectangle such that every point of every square is more than 1/2 units away from it to the centre of the circle. The maximum area A around each square in which the centre of the circle cannot be located is the area of the figure F formed by

(a) the square;

(b) four rectangles of dimensions $1 \times \frac{1}{2}$ external to the sides of the square;

(c) four quarters of circles with radius 1/2 units external to the square with and centres at the vertices of the square.

Hence, $A = 1 + \frac{1}{2} \cdot 1 \cdot 4 + (\frac{1}{2})^2\pi = 3 + \frac{\pi}{4}$. As there are 120 squares, the sum of all such areas within the rectangle does not exceed $120 \cdot (3 + \frac{\pi}{4}) < 455$.

As the circle should be placed inside of the rectangle, its centre cannot be less than 1/2 units away from the rectangle's sides, *i.e.*, it can be only in the rectangle with sides 19 and 24 units of length, whose sides are parallel to the rectangle's sides on the distance $1/2$ units from them. The area of this rectangle is $19 \times 24 = 456$. But $456 - 455 > 0$, so at least one point is not covered by any of the 120 figures F described above.. This point can be the centre of a circle of diameter 1 lying within the rectangle and having no point in common with any of the squares.

11. Each of nine lines divides a square into two quadrilaterals, such that the ratio of their area is 2:3. Prove that at least three of these lines are concurrent.

Solution. [Miranda Holmes] Since the lines divide the square into two quadrilaterals, they cut opposite sides of the square. Let the vertices of the square be A, B, C, D (counterclockwise), and let one of the lines intersect AB at M and CD at N. We can represent these points in an appropriate coordinate plane as $A(0,0)$, $B(1,0)$, $C(1,1)$, $D(0,1)$, $M(m,0)$, $N(n,1)$.

Let $[AMND] : [MBCN] = 2 : 3$. Then $[AMND] = (m+n)/2 = 2/5$, because the area of the whole square is 1. The midpoint of MN is the point $S(\frac{1}{2}(m+n), \frac{1}{2}) = S(\frac{2}{5}, \frac{1}{2})$, which does not depend on the points of intersection of M and N and, hence, is the same for all such lines. So, each line which divides the square into two quadrilaterals in this way, must go through the point S. Because of the symmetry, there are three other possible points in the square $(\frac{3}{5}, \frac{1}{2})$, $(\frac{1}{2}, \frac{2}{5})$, $(\frac{1}{2}, \frac{3}{5})$, and each of the given 9 points must pass through one of them. Applying the Pigeonhole Principle for 9 lines and 4 points, we find that at least three of the lines must pass through the same point, and because of that, they are concurrent.

12. Each vertex of a regular 100-sided polygon is marked with a number chosen from among the natural numbers $1, 2, 3, \cdots, 49$. Prove that there are four vertices (which we can denote as A, B, C, D with respective numbers a, b, c, d) such that $ABCD$ is a rectangle, the points A and B are two adjacent vertices of the rectangle and $a + b = c + d$.

Solution. Since the given polygon is regular, it can be inscribed in a circle. There are exactly 50 diagonals of the polygon which pass through the centre of the circle. As they are diameters, they are of equal length. Consider the positive differences of two vertices which are endpoints of the same diagonal. Since the mark numbers are from among $1, 2, \cdots, 49$, the range of the differences is between 0 and 48. So, we have 49 possible values for 50 differences. Hence, there are at least two diagonals with the same difference. Without loss of generality, denote these diagonals as as AC and BD and suppose that $a \geq c$ and $d \geq b$. Then $a - c = d - b$, so that $a + b = c + d$. The quadrilateral $ABCD$ has two diagonals of equal length and with the same midpoint, so it is a rectangle, which satisfies all of the required conditions.