PROBLEMS FOR DECEMBER

Please send your solutions to
Professor E.J. Barbeau
Department of Mathematics
University of Toronto
Toronto, ON M5S 3G3
no later than **January 31, 2002**.

Note. The *incentre* of a triangle is the centre of the inscribed circle that touches all three sides. A set is *connected* if, given two points in the set, it is possible to trace a continuous path from one to the other without leaving the set.

121. Let n be an integer exceeding 1. Let $a_1, a_2, \cdots, a_n$ be posive real numbers and $b_1, b_2, \cdots, b_n$ be arbitrary real numbers for which

$$\sum_{i \neq j} a_i b_j = 0 \ .$$

Prove that $\sum_{i \neq j} b_i b_j < 0$.

122. Determine all functions f from the real numbers to the real numbers that satisfy

$$f(f(x) + y) = f(x^2 - y) + 4f(x)y$$

for any real numbers x, y.

123. Let a and b be the lengths of two opposite edges of a tetrahedron which are mutually perpendicular and distant d apart. Determine the volume of the tetrahedron.

124. Prove that

$$\frac{(1^4 + \frac{1}{4})(3^4 + \frac{1}{4})(5^4 + \frac{1}{4}) \cdots (11^4 + \frac{1}{4})}{(2^4 + \frac{1}{4})(4^4 + \frac{1}{4})(6^4 + \frac{1}{4}) \cdots (12^4 + \frac{1}{4})} = \frac{1}{313} \ .$$

125. Determine the set of complex numbers z which satisfy

$$\text{Im } (z^4) = (\text{Re } (z^2))^2 \ ,$$

and sketch this set in the complex plane. (*Note:* Im and Re refer respectively to the imaginary and real parts.)

126. Let n be a positive integer exceeding 1, and let n circles (*i.e.*, circumferences) of radius 1 be given in the plane such that no two of them are tangent and the subset of the plane formed by the union of them is connected. Prove that the number of points that belong to at least two of these circles is at least n.