PROBLEMS FOR FEBRUARY

Solutions should be submitted to
Prof. E.J. Barbeau
Department of Mathematics
University of Toronto
Toronto, ON M5S 3G3
no later than **March 31, 2001**

61. Let $S = 1!2!3! \cdots 99!100!$ (the product of the first 100 factorials). Prove that there exists an integer k for which $1 \leq k \leq 100$ and $S/k!$ is a perfect square. Is k unique? (*Optional:* Is it possible to find such a number k that exceeds 100?)

62. Let n be a positive integer. Show that, with three exceptions, $n! + 1$ has at least one prime divisor that exceeds $n + 1$.

63. Let n be a positive integer and k a nonnegative integer. Prove that

$$n! = (n+k)^n - \binom{n}{1}(n+k-1)^n + \binom{n}{2}(n+k-2)^n - \cdots \pm \binom{n}{n}k^n \ .$$

64. Let M be a point in the interior of triangle ABC, and suppose that D, E, F are points on the respective side BC, CA, AB. Suppose AD, BE and CF all pass through M. (In technical terms, they are *cevians.*) Suppose that the areas and the perimeters of the triangles BMD, CME, AMF are equal. Prove that triangle ABC must be equilateral.

65. Suppose that XTY is a straight line and that TU and TV are two rays emanating from T for which $\angle XTU = \angle UTV = \angle VTY = 60°$. Suppose that P, Q and R are respective points on the rays TY, TU and TV for which $PQ = PR$. Prove that $\angle QPR = 60°$.

66. (a) Let $ABCD$ be a square and let E be an arbitrary point on the side CD. Suppose that P is a point on the diagonal AC for which $EP \perp AC$ and that Q is a point on AE produced for which $CQ \perp AE$. Prove that B, P, Q are collinear.

    (b) Does the result hold if the hypothesis is weakened to require only that $ABCD$ is a rectangle?