PROBLEMS FOR JANUARY

Solutions should be submitted to
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no later than **February 28, 2001**

55. A textbook problem has the following form: *A man is standing in a line in front of a movie theatre. The fraction x of the line is in front of him, and the fraction y of the line is behind him, where x and y are rational numbers written in lowest terms. How many people are there in the line?* Prove that, if the problem has an answer, then that answer must be the least common multiple of the denominators of x and y.

56. Let n be a positive integer and let $x_1, x_2, \cdots, x_n$ be integers for which

$$x_1^2 + x_2^2 + \cdots + x_n^2 + n^3 \leq (2n-1)(x_1 + x_2 + \cdots + x_n) + n^2 \ .$$

Show that

(a) $x_1, x_2, \cdots, x_n$ are all nonnegative;

(b) $x_1 + x_2 + \cdots + x_n + n + 1$ is not a perfect square.

57. Let $ABCD$ be a rectangle and let E be a point in the diagonal BD with $\angle DAE = 15°$. Let F be a point in AB with $EF \perp AB$. It is known that $EF = \frac{1}{2}AB$ and $AD = a$. Find the measure of the angle $\angle EAC$ and the length of the segment EC.

58. Find integers a, b, c such that $a \neq 0$ and the quadratic function $f(x) = ax^2 + bx + c$ satisfies

$$f(f(1)) = f(f(2)) = f(f(3)) \ .$$

59. Let $ABCD$ be a concyclic quadrilateral. Prove that

$$|AC - BD| \leq |AB - CD| \ .$$

60. Let $n \geq 2$ be an integer and $M = \{1, 2, \cdots, n\}$. For every integer k with $1 \leq k \leq n-1$, let

$$x_k = \sum \{\min\ A + \max\ A : A \subseteq M, A \text{ has } k \text{ elements}\}$$

where $\min A$ is the smallest and $\max A$ is the largest number in A. Determine $\sum_{k=1}^{n} (-1)^{k-1} x_k$.