Problems

Please send solutions to
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no later than December 31, 2001. Please make sure that your name, email address and complete mailing address are on the first page.

115. Let U be a set of n distinct real numbers and let V be the set of all sums of distinct pairs of them, *i.e.*,

$$V = \{x + y : x, y \in U, x \neq y\} .$$

What is the smallest possible number of distinct elements that V can contain?

116. Prove that the equation

$$x^4 + 5x^3 + 6x^2 - 4x - 16 = 0$$

has exactly two real solutions.

117. Let a be a real number. Solve the equation

$$(a-1)\left(\frac{1}{\sin x} + \frac{1}{\cos x} + \frac{1}{\sin x \cos x}\right) = 2 .$$

118. Let a, b, c be nonnegative real numbers. Prove that

$$a^2(b + c - a) + b^2(c + a - b) + c^2(a + b - c) \leq 3abc .$$

When does equality hold?

119. The medians of a triangle ABC intersect in G. Prove that

$$|AB|^2 + |BC|^2 + |CA|^2 = 3(|GA|^2 + |GB|^2 + |GC|^2) .$$

120. Determine all pairs of nonnull vectors $\mathbf{x}$, $\mathbf{y}$ for which the following sequence $\{a_n : n = 1, 2, \cdots\}$ is (a) increasing, (b) decreasing, where

$$a_n = |\mathbf{x} - n\mathbf{y}| .$$