## Solutions

73. Solve the equation:

$$\left(\sqrt{2+\sqrt{2}}\right)^x + \left(\sqrt{2-\sqrt{2}}\right)^x = 2^x \ .$$

*Solution 1.* By inspection, we find that $x = 2$ satisfies the equation. We show that no other value of $x$ does so. Observe that $\sqrt{2-\sqrt{2}} < 1$. When $x < 0$, the second term of the left side exceeds 1 while the right side is less than 1, so the equation is not satisfied. Henceforth, let $x \geq 0$ and let

$$f(x) \equiv 2^x - \left(\sqrt{2+\sqrt{2}}\right)^x$$

and $g(x) = (\sqrt{2-\sqrt{2}})^x$.

Note that, if $a > b > 1$, then $a^x - b^x = b^x((a/b)^x - 1)$ is an increasing function of $x$. Thus, $f(x)$ is increasing and $g(x)$ is decreasing as $x$ increases. If $0 \leq x < 2$, then $f(x) < f(2) = g(2) < g(x)$, while if $x > 2$, then $f(x) > f(2) = g(2) > g(x)$. The desired result follows.

*Solution 2.* The equation can be rewritten in the form

$$f(y) \equiv a^y + (1-a)^y = 1$$

where $2y = x$ and $a = \frac{1}{4}(2+\sqrt{2})$. Note that $0 < a < 1$, so that each term is a strictly decreasing function of $y$. Thus, $f(y)$ assumes each of its values at most once, and since $f(1) = 1$, we find that $x = 2$ is the only solution.

*Solution 3.* Observe that

$$1 + \frac{\sqrt{2}}{2} = 1 + \cos\frac{\pi}{4} = 2\cos^2\frac{\pi}{8}$$

and

$$1 - \frac{\sqrt{2}}{2} = 1 - \sin\frac{\pi}{4} = 2\sin^2\frac{\pi}{8} \ .$$

The equation becomes

$$\left(\cos\frac{\pi}{8}\right)^x + \left(\sin\frac{\pi}{8}\right)^x = 1 \ .$$

This holds for $x = 2$. If $x > 2$, then $x - 2 > 0$ and so

$$\left(\cos\frac{\pi}{8}\right)^x = \left(\cos\frac{\pi}{8}\right)^{x-2}\left(\cos\frac{\pi}{8}\right)^2 < \left(\cos\frac{\pi}{8}\right)^2$$

with a similar inequality for the sine function. Thus, when $x > 2$, the left side is less than 1. Similarly, it can be shown that when $x < 2$, the left side exceeds 1. Hence the unique solution is $x = 2$.

*Comment.* Generally, the solutions involved a function similar to that used in Solution 2, and it was shown that it was impossible for there to be more than one solution. Some students came up with the use of trigonometry as in Solution 3.

74. Prove that among any group of $n+2$ natural numbers, there can be found two numbers so that their sum or their difference is divisible by $2n$.

*Solution 1.* For $0 \leq k \leq n$, let $S_k$ be the subset of numbers $x$ among the $n+2$ numbers for which $x$ differs from either $k$ or $2n-k$ by a multiple of $2n$. Since there are $n+2$ numbers and only $n+1$ subsets, the Pigeonhole Principle provides that some subset must contain at least two numbers $u$ and $v$, say. Either

$u$ and $v$ both leave the same remainder upon division by $2n$ and so differ by a multiple of $2n$, or else one of them differs from $k$ by a multiple of $2n$ while the other differs from $2n-k$ by a multiple of $2n$. In the latter case, $u+v$ is a multiple of $2n$.

*Solution 2.* [A. Fink] Consider an arbitrary set of $n+2$ natural numbers. If any two are congruent modulo $2n$, then their difference is divisible by $2n$ and the result follows. Suppose otherwise, that all numbers have distinct residues modulo $2n$. Apportion these residues into the $n+1$ sets: $\{0\}$, $\{1, 2n-1\}$, $\{2, 2n-2\}$, $\cdots$, $\{n-1, n+1\}$, $\{n\}$. Since there are $n+2$ numbers, at least one of these sets must contain two residues, and so the two numbers involved must sum to a multiple of $2n$.

75. Three consecutive natural numbers, larger than 3, represent the lengths of the sides of a triangle. The area of the triangle is also a natural number.

(a) Prove that one of the altitudes "cuts" the triangle into two triangles, whose side lengths are natural numbers.

(b) The altitude identified in (a) divides the side which is perpendicular to it into two segments. Find the difference between the lengths of these segments.

*Solution 1.* Let the side lengths be $x-1$, $x$, $x+1$. By Heron's formula, the area $A$ of the triangle is given by

$$A^2 = \frac{3}{16}x^2(x+2)(x-2) = \frac{(3(x^2-4))x^2}{16} .$$

Since $A$ is an integer, $x$ must be even and $3(x^2-4)$ must be the square of a multiple of $2 \times 3 = 6$. Hence for some integer $y$, we have that $3(x^2-4) = (6y)^2 = 36y^2$ or $x^2 - 12y^2 = 4$. (*Comment.* This is a Pell's equation and it has infinitely many solutions $(x, y) = (x_n, y_n)$ given by $(x_n, y_n) = 2(7+2\sqrt{12})^n$ and $(4+\sqrt{12})(7+2\sqrt{12})^n$.

(a) The area $A$ of the triangle is $\frac{1}{4}(6xy) = \frac{3xy}{2}$ where $x$ is even and $x^2 - 12y^2 = 4$. The altitude to the side of length $x$ is $2A/x = 3y$, an integer. This is the desired altitude.

(b) The triangle is subdivided by the altitude in (a) into two right triangles whose hypotenuses have lengths $x-1$ and $x+1$. Hence, the side of length $x$ is split into two parts of lengths

$$\sqrt{(x-1)^2-(3y)^2} = \sqrt{x^2-2x+1-9y^2} = \sqrt{(x^2/4)-2x+4} = \frac{1}{2}(x-4)$$

and

$$\sqrt{(x+1)^2-(3y)^2} = \sqrt{x^2+2x+1-9y^2} = \sqrt{(x^2/4)+2x+4} = \frac{1}{2}(x+4) .$$

The difference between the lengths of these segments is 4. (Note that the sum is $x$. as expected.) (*Exercise.* Give some numerical examples.)

*Solution 2.* [L. Tchourakov] With the above notation, we find that $A = x\sqrt{3(x^2-4)}/4$, so that the length of the altitude to the side of length $x$ is $\frac{1}{2}\sqrt{3(x^2-4)}$. If $x$ were odd, then the numerator of the fraction for $A$ would be odd and $A$ not an integer. Hence $x$ is even, and so is the altitude. Let $u$ be one of the two parts of the side cut off by the altitude. By the pythagorean theorem,

$$u^2 = (x+1)^2 - \frac{3(x^2-4)}{4} = \frac{(x+4)^2}{4} ,$$

so that $u = 2 + x/2$. Since $x$ is even, $u$ is an integer. The altitude cuts the side into parts of length $u$ and $x - u = u - 4$, and so (b) follows.

76. Solve the system of equations:

$$\log x + \frac{\log(xy^8)}{\log^2 x + \log^2 y} = 2 ,$$

$$\log y + \frac{\log(x^8/y)}{\log^2 x + \log^2 y} = 0 \ .$$

(The logarithms are taken to base 10.)

*Solution 1.* Let $u = \log x$, $v = \log y$ and $w = u^2 + v^2$. Note that $w$ is nonzero. The equations become

$$u + (u + 8v)/w = 2 \qquad \text{and} \qquad v + (8u - v)/w = 0 \ .$$

Squaring and adding the equations $u + 8v = (2 - u)w$ and $8u - v = -vw$ yields $65(u^2 + v^2) = (4 - 4w + u^2 + v^2)w^2$, or $65 = (4 - 4u + w)w$. We can also write the system as

$$(w + 1)u + 8v = 2w$$
$$8u + (w - 1)v = 0 \ ,$$

which can be solved to yield

$$u = \frac{2w(w-1)}{w^2 - 65} \qquad v = \frac{16w}{65 - w^2} \ .$$

Hence

$$w = u^2 + v^2 = \frac{4w^2(w-1)^2 + 256w^2}{(65 - w^2)}$$
$$\Longrightarrow (65 - w^2)^2 = 4w(w-1)^2 + 256w$$
$$\Longrightarrow$$
$$0 = w^4 - 4w^3 - 122w^2 - 260w + 65^2$$
$$= (w - 13)(w - 5)[(w + 7)^2 + 4^2] \ .$$

The two relevant solutions are $w = 13$ and $w = 5$.

When $w = 13$, $17 - 4u = 5$, which leads to $(u, v) = (3, -2)$. When $w = 5$, $9 - 4u = 13$, so that $(u, v) = (-1, 2)$. The desired solutions are $(x, y) = (10^3, 10^{-2}), (10^{-1}, 10^2)$.

*Solution 2.* With the same notation as (1), we can write the given equations in terms of $u$ and $v$. Multiply the first equation by $u$ and the second by $v$ and add them to obtain the equation $2uv + 8 = 2v$, whereupon $u = 1 - (4/v)$. Eliminating $u$ from the second equation yields $v^4 - 16 = 0$, whereupon $v = 2$ or $v = -2$. The remaining part of the solution is easily completed.

77. $n$ points are chosen from the circumference or the interior of a regular hexagon with sides of unit length, so that the distance between any two of them is **not** less than $\sqrt{2}$. What is the largest natural number $n$ for which this is possible?

78. A truck travelled from town $A$ to town $B$ over several days. During the first day, it covered $1/n$ of the total distance, where $n$ is a natural number. During the second day, it travelled $1/m$ of the remaining distance, where $m$ is a natural number. During the third day, it travelled $1/n$ of the distance remaining after the second day, and during the fourth day, $1/m$ of the distance remaining after the third day. Find the values of $m$ and $n$ if it is known that, by the end of the fourth day, the truck had travelled $3/4$ of the distance between $A$ and $B$. (Without loss of generality, assume that $m < n$.)

*Solution.* [R. Furmaniak, J. Rin] Let $d$ be the distance remaining at the beginning of a two-day period. The distance remaining at the end of the period is

$$d - \left[\frac{d}{n} + \frac{1}{m}\left(d - \frac{d}{n}\right)\right] = d\left(1 - \frac{1}{m}\right)\left(1 - \frac{1}{n}\right) \ .$$

Thus, every two days the remaining distance is reduced by a factor of $r = (1 - (1/m))(1 - (1/n))$. (Note that this is symmetric in $m$ and $n$.) After four days, the distance remaining is reduced by a factor of $r^2$; it is given in the problem that this is $1/4$. Hence $r = 1/2$.

Hence

$$\begin{aligned}\frac{1}{2} &= \left(1-\frac{1}{m}\right)\left(1-\frac{1}{n}\right)\\ &\Longrightarrow mn = 2(m-1)(n-1)\\ &\Longrightarrow 0 = mn - 2(m+n) + 2\\ &\Longrightarrow 2 = (m-2)(n-2)\ .\end{aligned}$$

Since $m$ and $n$ are positive integers with $m < n$, we must have $n-2=2$ and $m-2=1$, *i.e.*, $(m,n)=(3,4)$.