Solutions and comments.

Notes. A *rectangular hyperbola* is an hyperbola whose asymmptotes are at right angles.

97. A triangle has its three vertices on a rectangular hyperbola. Prove that its orthocentre also lies on the hyperbola.

Solution 1. A rectangular hyperbola can be represented as the locus of the equation $xy = 1$. Let the three vertices of the triangle be at $(a, 1/a)$, $(b, 1/b)$, $(c, 1/c)$. The altitude to the points $(c, 1/c)$ has slope $-(a-b)/(a^{-1} - b^{-1}) = ab$ and its equation is $y = abx + (1/c) - abc$. The altitude to the point $(a, 1/a)$ has equation $y = bcx + (1/a) - abc$. These two lines intersect in the point $(-1/abc, -abc)$ and the result follows.

Solution 2. [R. Barrington Leigh] Suppose that the equation of the rectangular hyperbola is $xy = 1$. Let the three vertices be at (x_i, y_i) $(i = 1, 2, 3)$, and let the orthocentre be at (x_0, y_0). Then

$$(x_1 - x_2)(x_0 - x_3) = -(y_1 - y_2)(y_0 - y_3)$$

and

$$(x_1 - x_3)(x_0 - x_2) = -(y_1 - y_3)(y_0 - y_2) \ .$$

Cross-multiplying these equations yields that

$$(x_1 - x_2)(y_1 - y_3)(x_0 - x_3)(y_0 - y_2) = (x_1 - x_3)(y_1 - y_2)(x_0 - x_2)(y_0 - y_3) \ ,$$

whence

$$(1 - x_1y_3 - x_2y_1 + x_2y_3)(x_0y_0 - x_0y_2 - x_3y_0 + x_3y_2) = (1 - x_1y_2 - x_3y_1 + x_3y_2)(x_0y_0 - x_0y_3 - x_2y_0 + x_2y_3) \ .$$

Collecting up the terms in x_0y_0, x_0, y_0, and the rest, and simplifying, yields that $x_0y_0 = 1$, as desired.

98. Let $a_1, a_2, \cdots, a_{n+1}, b_1, b_2, \cdots, b_n$ be nonnegative real numbers for which
(i) $a_1 \geq a_2 \geq \cdots \geq a_{n+1} = 0$,
(ii) $0 \leq b_k \leq 1$ for $k = 1, 2, \cdots, n$.
Suppose that $m = \lfloor b_1 + b_2 + \cdots + b_n \rfloor + 1$. Prove that

$$\sum_{k=1}^{n} a_k b_k \leq \sum_{k=1}^{m} a_k \ .$$

Solution. Note that $m - 1 \leq b_1 + b_2 + \cdots + b_m < m$. We have that

$$\begin{aligned} a_1b_1 + a_2b_2 + \cdots + a_mb_m + a_{m+1}b_{m+1} + \cdots + a_nb_n & \\ \leq a_1b_1 + a_2b_2 + \cdots + a_mb_m + a_m(b_{m+1} + b_{m+2} + \cdots + b_n) & \\ < a_1b_1 + a_2b_2 + \cdots + a_mb_m + a_m(m - b_1 - b_2 - \cdots - b_m) & \\ = a_1b_1 + a_2b_2 + \cdots + a_mb_m + a_m(1 - b_1) + a_m(1 - b_2) + \cdots + a_m(1 - b_m) & \\ \leq a_1b_1 + a_2b_2 + \cdots + a_mb_m + a_1(1 - b_1) + a_2(1 - b_2) + \cdots + a_m(1 - b_m) & \\ = a_1 + a_2 + \cdots + a_m \ . & \end{aligned}$$

99. Let E and F be respective points on sides AB and BC of a triangle ABC for which $AE = CF$. The circle passing through the points B, C, E and the circle passing through the points A, B, F intersect at B and D. Prove that BD is the bisector of angle ABC.

Solution 1. Because of the concyclic quadrilaterals, $\angle DEA = 180^\circ - \angle BED = \angle DCF$ and $\angle DFC = 180^\circ - \angle DFB = \angle DAB$. Since, also, $AE = CF$, $\Delta DAE \equiv \Delta DFC$ (ASA) so that $AD = DF$. In the circle through $ABFD$, the equal chords AD and DF subtend equal angles ABD and FBD at the circumference. The result follows.

Solution 2. $\angle CDF = \angle CDE - \angle FDE = 180° - \angle ABC - \angle FDE = \angle FDA - \angle FDE = \angle EDA$ and $\angle AED = 180° - \angle BED = \angle BCD = \angle FCD$. Since $AE = CF$, $\Delta EAD \equiv \Delta CFD$ (ASA). The altitude from D to AE is equal to the altitude from D to FC, and so D must be on the bisector of $\angle ABC$.

Solution 3. Let B be the point $(0, -1)$ and D the point $(0, 1)$. The centres of both circles are on the right bisector of BD, namely the $x-$axis. Let the two circles have equations $(x - a)^2 + y^2 = a^2 + 1$ and $(x - b)^2 + y^2 = b^2 + 1$. Suppose that $y = mx - 1$ is a line through B; this line intersects the circle of equation $(x - a)^2 + y^2 = a^2 + 1$ in the point

$$\left(\frac{2(m + a)}{m^2 + 1}, \frac{m^2 + 2am - 1}{m^2 + 1}\right)$$

and the circle of equation $(x - b)^2 + y^2 = b^2 + 1$ in the point

$$\left(\frac{2(m + b)}{m^2 + 1}, \frac{m^2 + 2bm - 1}{m^2 + 1}\right)$$

The distance between these two points is the square root of

$$\left[\frac{2(a - b)}{m^2 + 1}\right]^2 + \left[\frac{2m(a - b)}{m^2 + 1}\right]^2 = \frac{4(a - b)^2(1 + m^2)}{(m^2 + 1)^2} = \frac{4(a - b)^2}{m^2 + 1} .$$

Now suppose that the side AB of the triangle has equation $y = m_1x - 1$ and the side BC the equation $y = m_2x - 1$, so that (A, E) and (C, F) are the pairs of points where the lines intersect the circles. Then, from the foregoing paragraph, we must have $m_1^2 + 1 = m_2^2 + 1$ or $0 = (m_1 - m_2)(m_1 + m_2)$. Since the sides are distinct, it follows that $m_1 = -m_2$ and so BD bisects $\angle ABC$.

100. If 10 equally spaced points around a circle are joined consecutively, a convex regular inscribed decagon P is obtained; if every third point is joined, a self-intersecting regular decagon Q is formed. Prove that the difference between the length of a side of Q and the length of a side of P is equal to the radius of the circle. [*With thanks to Ross Honsberger.*]

Solution 1. Let the decagon be $ABCDEFGHIJ$. Let BE and DI intersect at K and let AF and DI intersect at L. Observe that $AB \| DI \| EH$ and $BE \| AF \| HI$, so that $ABKL$ and $KIHE$ are parallelograms. Now AB is a side of P and HE is a side of Q, and the length of the segment IL is the difference of the lengths of $EH = IK$ and $AB = KL$. Since L, being the intersection of the diameters AF and DI, is the centre of the circle, the result follows.

Solution 2. [R. Barrington Leigh] Use the same notation as in Solution 1. Let O be the centre of P. Now, AB is an edge of P, AD is an edge of Q, DO is a radius of the circle and BG a diameter. Let AD and BO intersect at U. Identify in turn the angles $\angle DOU = 72°$, $\angle DAB = 36°$, $\angle ABU = 72°$, $\angle DUO = \angle BUA = 72°$, whence $AU = AB$, $DU = DO$ and $AD - AB = AD - AX = DX = DO$, as desired.

Solution 3. Label the vertices of P as in Solution 1. Let O be the centre of P, and V be a point on EB for which $EV = OE$. We have that $\angle AOB = 36°$, $\angle DOB = \angle OBA = 72°$, $\angle BOE = 108°$ and $\angle OEB = \angle OBE = 36°$. Also, $\angle EOV = \angle EVO = 72°$ and $OE = EV = OA = OB$. Hence, $\Delta DAB = \Delta EVO$ (SAS), so that $OV = AB$. Since $\angle BVO = 108°$ and $\angle BOV = 36°$, $\angle OBV = 36°$, and so $BV = OV = AB$. Hence $BE - AB = EV + BV - AB = EV = OE$, the radius.

Solution 4. Let the circumcircle of P and Q have radius 1. A side of P is the base of an isosceles triangle with equal sides 1 and apex angle $36°$, so its length is $2\sin 18°$. Likewise, the length of a side of Q is $2\sin 54°$. The difference between these is

$$2\sin 54° - 2\sin 18° = 2\cos 36° - 2\cos 72° = 2t - 2(2t^2 - 1) = 2 + 2t - 4t^2$$

where $t = \cos 36°$. Now

$$t = \cos 36° = -\cos 144° = 1 - 2\cos^2 72°$$
$$= 1 - 2(2t^2 - 1)^2 = -8t^4 + 8t^2 - 1 \ ,$$

so that

$$0 = 8t^4 - 8t^2 + t + 1 = (t+1)(8t^3 - 8t^2 + 1)$$
$$= (t+1)(2t-1)(4t^2 - 2t - 1) \ .$$

Since t is equal to neither -1 nor $\frac{1}{2}$, we must have that $4t^2 - 2t = 1$. Hence

$$2\sin 54° - 2\sin 18° = 2 - (4t^2 - 2t) = 1 \ ,$$

the radius of the circle.

101. Let a, b, u, v be nonnegative. Suppose that $a^5 + b^5 \leq 1$ and $u^5 + v^5 \leq 1$. Prove that

$$a^2u^3 + b^2v^3 \leq 1 \ .$$

[*With thanks to Ross Honsberger.*]

Solution. By the arithmetic-geometric means inequality, we have that

$$\frac{2a^5 + 3u^5}{5} = \frac{a^5 + a^5 + u^5 + u^5 + u^5}{5} \geq \sqrt[5]{a^{10}u^{15}} = a^2u^3$$

and, similarly,

$$\frac{2b^5 + 3v^5}{5} \geq b^2v^3 \ .$$

Adding these two inequalities yields the result.

102. Prove that there exists a tetrahedron $ABCD$, all of whose faces are similar right triangles, each face having acute angles at A and B. Determine which of the edges of the tetrahedron is largest and which is smallest, and find the ratio of their lengths.

Solution 1. Begin with AB, a side of length 1. Now construct a rectangle $ACBD$ with diagonal AB, so that $|AC| = |BD| = s < t = |AD| = |BC|$. The requisite values of s and t will be determined in due course. We want to show that we can fold up D and C from the plane in which AB lies (like folding up the wings of a butterfly) in such a way that we can obtain the desired tetrahedron.

When the triangles ADB and ACB lie flat, we see that C and D are distance 1 apart. Suppose that, when we have folded up C and D to get the required tetrahedron, they are distance r apart. Then ACD should be a right triangle similar to ABC. The hypotenuse of ΔACD cannot be AC as $AC < AD$. Nor can it be CD, for then, we would have $AD = BC$, $AC = AC$, and CD would have to have length 1, possible only when $ABCD$ is coplanar. So the hypotenuse must be AD. The similarity of ΔADC and ΔABC would require that

$$1 : t : s = t : s : r$$

where $r = |CD|$. Thus, $1/t = t/s$ or $s = t^2$ and $t/s = s/r$ or $r = s^2/t = t^3$. So we must fold C and D until they are distance t^3 apart.

Is this possible? Since ΔACB is right, $1 = t^2 + s^2 = t^2 + t^4$, whence $s = t^2 = \frac{1}{2}(-1 + \sqrt{5}) < 1$. Hence $r < 1$. To arrange that we can make the distance between C and D equal to r, we must show that r exceeds the minimum possible distance between C and D, which occurs when ΔADB is folded flat partially covering ΔACB. Suppose this has been done, with $ABCD$ coplanar and C, D both on the same side of AB. Let P and Q be the respective feet of the perpendiculars to AB from C and D. Then

$$|CP| = |DQ| = t^3 \ , \quad |AP| = |QB| = t^4 \ , \quad |AQ| = |PB| = t^2 \ ,$$

and

$$|CD| = |PQ| = t^2 - t^4 = (t^4 + t^6) - t^4 = t^6 < t^3 \ .$$

When C and D are located, we have $|AB| = 1$, $|AD| = |BC| = t$, $|AC| = |BD| = t^2$ and $|CD| = t^3$. Since all faces of the tetrahedron $ABCD$ have sides in the ratio $1 : t : t^2$, all are similar right triangles and $AB : CD = 1 : t^3$.

Solution 2. Let $\alpha = \angle CAB$ and $|AB| = 1$. By the condition on the acute angles of triangles ACB and ACD, $\angle ACB = \angle ADB = 90^\circ$, so that the triangles ΔACD and ΔADB, being similar and sharing a hypotenuse, are congruent.

Suppose, if possible, that $\angle BAD = \alpha$. Then $AC = AD$ and so ΔACD must be isosceles with its right angle at A, contrary to hypothesis. So, $\angle ABD = \alpha$ and $|BD| = |AC| = \cos\alpha$, $|AD| = |BC| = \sin\alpha$.

Consider ΔACD. Suppose that $\angle ACD = 90^\circ$. If $\angle DAC = \alpha$, then $\Delta ABC \equiv \Delta ADC$ and $1 = |AB| = |AD| = \sin\alpha$, yielding a contradiction. Hence $\angle ADC = \alpha$, $|AD| = |AC|/\sin\alpha = \cos\alpha/\sin\alpha$ and $|CD| = |AC|\cot\alpha = \cos^2\alpha/\sin\alpha$. Hence, looking at $|AD|$, we have that

$$\frac{\cos\alpha}{\sin\alpha} = \sin\alpha \Longrightarrow 0 = \cos\alpha - \sin^2\alpha = \cos^2\alpha + \cos\alpha - 1 \ .$$

Therefore, $\cos\alpha = \frac{1}{2}(\sqrt{5} - 1)$ and $\sin^2\alpha = \cos\alpha$.

Observe that $|BC|\sin\alpha = \sin^2\alpha = \cos\alpha = |BD|$ and $|BC|\cos\alpha = \sin\alpha\cos\alpha = \cos^2\alpha/\sin\alpha = |CD|$, so that triangle BCD is right with $\angle CDB = 90^\circ$ and similar to the other three faces.

We need to check that this set-up is feasible. Using spatial coordinates, take

$$C \sim (0,0,0) \qquad A \sim (0, \cos\alpha, 0) \qquad B \sim (\sin\alpha, 0, 0) \ .$$

Since $\angle ACD = 90^\circ$, D lies in the plane $y = 0$ and so has coordinates of the form $(x, 0, z)$. Since $\angle CDB = 90^\circ$, $CD \perp DB$, so that

$$0 = (x, 0, z) \cdot (x - \sin\alpha, 0, z) - x^2 + z^2 - x\sin\alpha \ ,$$

Now $|CD| = \cos\alpha\sin\alpha$ forces $\cos^2\alpha\sin^2\alpha = x^2 + z^2$. Hence

$$x\sin\alpha = \cos^2\alpha\sin^2\alpha \Longrightarrow x = \cos^2\alpha\sin\alpha \ .$$

Therefore

$$z^2 = (\cos^2\alpha - \cos^4\alpha)\sin^2\alpha = \cos^2\alpha\sin^4\alpha \Longrightarrow z = \cos\alpha\sin^2\alpha \ ,$$

Hence $D \sim (\cos^2\alpha\sin\alpha, 0, \cos\alpha\sin^2\alpha)$.

Thus, letting $\sin\alpha = t = \frac{1}{2}(\sqrt{5} - 1)$, we have $A \sim (0, t^2, 0)$, $B \sim (t, 0, 0)$, $C \sim (0,0,0)$, $D \sim (t^5, 0, t^4)$ with $t^4 + t^2 - 1 = 0$, and $|AB| = 1$, $|AD| = |BC| = t$, $|BD| = |AC| = t^2$ and $|CD| = t^3$. [*Exercise:* Check that the coordinates give the required distances and similar right triangles.] The ratio of largest to smallest edges is $1 : t^3 = 1 : [\frac{1}{2}(\sqrt{5} - 1)]^{3/2} = 1 : \sqrt{2 + \sqrt{5}}$.

We need to dispose of the other possibilities for ΔACD. By the given condition, $\angle DAC \neq 90^\circ$. If $\angle ADC = 90^\circ$, then we have essentially the same situation as before with the roles of α and its complement, and of C and D switched.

Comment. Another way in that was used by several solvers was to note that there are four right angles involved among the four sides, and that at most three angles can occur at a given vertex of the tetrahedron. It is straightforward to argue that it is not possible to have three of the right angles at either C or D. Since all right angles occur at these two vertices, then there must be two at each. As an exercise, you might want to complete the argument from this beginning.