Solutions

Note. The *incentre* of a triangle is the centre of the inscribed circle that touches all three sides. A set is *connected* if, given two points in the set, it is possible to trace a continuous path from one to the other without leaving the set. $[X \cdots Z]$ refers to the area of the plane figure $X \cdots Z$.

121. Let n be an integer exceeding 1. Let $a_1, a_2, \cdots, a_n$ be posive real numbers and $b_1, b_2, \cdots, b_n$ be arbitrary real numbers for which

$$\sum_{i \neq j} a_i b_j = 0 \ .$$

Prove that $\sum_{i \neq j} b_i b_j < 0$.

Solution 1. For the result to hold, we need to assume that at least one of the b_i is nonzero. The condition is that

$$(a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n) = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \ .$$

Now

$$\begin{aligned} 2\sum_{i \neq j} b_i b_j &= (b_1 + b_2 + \cdots + b_n)^2 - (b_1^2 + b_2^2 + \cdots + b_n^2) \\ &= \frac{(a_1 b_1 + \cdots + a_n b_n)^2}{(a_1 + \cdots + a_n)^2} - (b_1^2 + \cdots + b_n^2) \\ &\leq \frac{(a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2)}{(a_1 + \cdots + a_n)^2} - (b_1^2 + \cdots + b_n^2) \\ &= \frac{(b_1^2 + \cdots + b_n^2)}{(a_1 + \cdots + a_n)^2}[(a_1^2 + \cdots + a_n^2) - (a_1 + \cdots + a_n)^2] < 0 \ , \end{aligned}$$

from which the desired result follows. The inequality is due to the Cauchy-Schwarz Inequality.

Solution 2. [R. Barrington Leigh] Suppose that not all the b_i vanish and that $b_1 \geq b_2 \geq \cdots \geq b_n$ (wolog). Since $\sum_{i \neq j} a_i b_j = 0$, not all the b_i have the same sign, and so $b_1 > 0 > b_n$. Wolog, we may assume that $B \equiv b_1 + b_2 + \cdots + b_n \geq 0$. (If $B < 0$, we can change the signs of all the b_i which alters neither the hypothesis nor the conclusion.) We have that

$$0 = a_1(b_1 - B) + a_2(b_2 - B) + \cdots + a_n(b_n - B) < (a_1 + a_2 + \cdots + a_n)(b_1 - B) \ ,$$

so that $b_1 > B$. Hence

$$2\sum_{i \neq j} b_i b_j = B^2 - \sum b_i^2 < B^2 - b_1^2 < 0 \ ,$$

as desired.

122. Determine all functions f from the real numbers to the real numbers that satisfy

$$f(f(x) + y) = f(x^2 - y) + 4f(x)y$$

for any real numbers x, y.

Solution 1. Let $y = \frac{1}{2}(x^2 - f(x))$. Then

$$f\left(\frac{f(x) + x^2}{2}\right) = f\left(\frac{f(x) + x^2}{2}\right) + 2f(x)[x^2 - f(x)] \ ,$$

from which it follows that, for each x, either $f(x) = 0$ or $f(x) = x^2$. [*Note:* this does **not** imply yet that the same option holds for all x.] In particular, $f(0) = 0$, so that $f(y) = f(-y)$ for all y.

Suppose that $f(c) = 0$. Then, for each real y, $f(y) = f(c^2 - y)$, whence $f(c^2) = f(0) = 0$. Thus, for each real y, $f(y) = f(c^4 - y)$. Suppose that $f(y) \neq 0$. Then

$$f(y) = y^2 \Longrightarrow y^2 = (c^2 - y)^2 = (c^4 - y)^2 \Longrightarrow 0 = c^2(c^2 - 2y) = c^4(c^4 - 2y) \ .$$

If c were nonzero, then we would have $c^2/2 = y = c^4/2$, so $c = 1$ and $y = \frac{1}{2}$. But then $f(-y) = f(y) = f(1 - y)$; substituting $y = -\frac{1}{2}$ yields $\frac{1}{4} = f(\frac{1}{2}) = f(\frac{3}{2})$, which is false. Hence $c = 0$. It follows that, either $f(x) \equiv 0$ (for all x) or else that $f(x) \equiv x^2$ (for all x). These solutions can (and should be) checked.

Solution 2. Let $y = x^2 - f(x)$. Then

$$f(x^2) = f(f(x) + x^2 - f(x)) = f(f(x)) + 4f(x)[x^2 - f(x)] \ .$$

Taking $y = 0$, we see that $f(f(x)) = f(x^2)$, so that $4f(x)[x^2 - f(x)] = 0$. Hence, for each x, either $f(x) = 0$ or $f(x) = x^2$.

Suppose, if possible, that there are two nonzero reals u and v for which $f(u) = 0$ and $f(v) = v^2$. Setting $(x, y) = (u, v)$ yields that $v^2 = f(u^2 - v)$. Since $v \neq 0$, we must have that

$$v^2 = f(u^2 - v) = u^4 - 2u^2v + v^2 \Rightarrow 0 = u^2(u^2 - 2v) \Rightarrow v = \frac{1}{2}u^2 \ .$$

This would mean that we could find only one such pair (u, v), which is false. Hence this case is not possible, so that, either $f(x) = 0$ for all x or else that $f(x) = x^2$ for all x.

Solution 3. From $(x, y) = (0, 0)$, we have that $f(f(0)) = f(0)$. From $(x, y) = (0, -f(0))$, we have that $f(0) = f(f(0)) - 4f(0)^2$, whence $f(0) = 0$. From $x = 0$, we have that $f(y) = f(-y)$ for all y. Finally, taking $y = x^2$ and $y = -f(x)$, we get

$$f(x^2 + f(x)) = f(0) + 4f(x)x^2 = f(x^2 + f(x)) - 4f(x)^2 + 4f(x)x^2 \ .$$

so that $0 = 4f(x)[x^2 - f(x)]$. We can finish as in the other solutions.

Solution 4. [R. Barrington Leigh] Taking $y = x - f(x)$ and then $y = x^2 - x$ yields that

$$\begin{aligned} f(x) &= f(f(x) + x - f(x)) = f(x^2 - x + f(x)) + 4f(x)(x - f(x)) \\ &= f(x^2 - (x^2 - x)) + 4f(x)(x^2 - x) + 4f(x)(x - f(x)) \\ &= f(x)[1 + 4x^2 - 4x + 4x - 4f(x)] = f(x) + 4f(x)[x^2 - f(x)] \ , \end{aligned}$$

so that for each x, either $f(x) = 0$ or $f(x) = x^2$. The solution can be completed as before.

123. Let a and b be the lengths of two opposite edges of a tetrahedron which are mutually perpendicular and distant d apart. Determine the volume of the tetrahedron.

Solution 1. Construct parallel planes distant d apart that contain the edges of lengths a and b. In the planes, congruent parallelograms can be constructed whose diagonals are of lengths a and b and right bisect each other, and each of which has an edge of the tetrahedron as a diagonal. Each parallelogram can be obtained from the other by a translation relating their centres, so the two parallelograms bound a prism with opposite faces distant d apart. The volume of this prism is $\frac{1}{2}abd$.

The prism is the disjoint union of the given tetrahedron and four tetrahedra, all of height d, two having as base a triangle with base a and height $\frac{1}{2}b$ and two having as base a triangle with base b and height $\frac{1}{2}a$. Each of these latter four tetrahedra have volume $\frac{1}{3}(\frac{1}{2} \cdot \frac{ab}{2})d = \frac{abd}{12}$. Hence, the volume of the given tetrahedron is

$$\frac{abd}{2} - 4\left(\frac{abd}{12}\right) = \frac{1}{6}(abd) \ .$$

Solution 2. Suppose that $ABCD$ is the tetrahedron with opposite edges AB of length a and CD of length b orthogonal and at distance d from each other.

Case (i). Suppose that AB and CD are oriented so that there are points E and F on AB and CD respectively for which EF is perpendicular to both AB and CD. Then $|EF| = d$ and $[ABF] = \frac{1}{2}ad$. The tetrahedron $ABCD$ is the union of the nonoverlapping tetrahedra $ABFC$ and $ABFD$, each with ΔABF as "base" and perpendicular height along CD. Hence the volume of $ABCD$ is equal to

$$\frac{1}{3}[ABF](|FC| + |FD|) = \frac{1}{3}\left(\frac{1}{2}ad\right)|CD| = \frac{1}{6}abd \ .$$

Case (ii). Suppose that E and F are on AB possibly produced and on CD produced, say, with EF perpendicular to AB and CD. Then we can argue in a way similar to that in Case (i) that the volume of $ABCD$ is equal to the volume of $ABFC$ less the volume of $ABFD$ to obtain the answer $(1/6)abd$.

Solution 3. [C. Lau; H. Lee] Let $ABCD$ be the given tetrahedron with $|BC| = a$ and $|AD| = b$. Suppose E lies on BC, possibly produced, with $AE \perp BC$. Then AD must lie in the plane containing AE and perpendicular to BC. Let F lie on AD produced with $EF \perp AD$. Note that $|EF| = d$. Let G be the foot of the perpendicular from D to AE produced. Then

$$[ADE] = \frac{1}{2}|AD||EF| = \frac{1}{2}bd = \frac{1}{2}|AE||GD| \ .$$

It follows that the volume of $ABCD$ is equal to

$$\frac{1}{3}[ABC]|GD| = \frac{1}{6}|AE||BC||GD| = \frac{1}{6}abd \ .$$

124. Prove that

$$\frac{(1^4 + \frac{1}{4})(3^4 + \frac{1}{4})(5^4 + \frac{1}{4}) \cdots (11^4 + \frac{1}{4})}{(2^4 + \frac{1}{4})(4^4 + \frac{1}{4})(6^4 + \frac{1}{4}) \cdots (12^4 + \frac{1}{4})} = \frac{1}{313} \ .$$

Solution. The left side can be written as

$$\prod\{4x^4 + 1 : x \ \text{ odd}, \quad 1 \le x \le 11\} \div \prod\{4x^4 + 1 : x \ \text{ even}, \quad 2 \le x \le 12\} \ .$$

Now

$$\begin{aligned} 4x^4 + 1 &= 4x^4 + 4x^2 + 1 - 4x^2 = (2x^2 + 1)^2 - (2x)^2 \\ &= (2x^2 - 2x + 1)(2x^2 + 2x + 1) = [(x-1)^2 + x^2][x^2 + (x+1)^2] \ . \end{aligned}$$

¿From this, we see that the left side is equal to

$$\frac{[1^2(1^2 + 2^2)][(2^2 + 3^2)(3^2 + 4^2)] \cdots [(10^2 + 11^2)(11^2 + 12^2)]}{[(1^2 + 2^2)(2^2 + 3^2)][(3^2 + 4^2)(4^2 + 5^2)] \cdots [(11^2 + 12^2)(12^2 + 13^2)]} = \frac{1^2}{12^2 + 13^2} = \frac{1}{313} \ .$$

Comment. In searching for factors, note that any common divisor of $4n^4 + 1$ and $4(n+1)^4 + 1$ must divide the difference

$$[(n^4 + 1)^4 - n^4] = [(n+1)^2 - n^2][(n+1)^2 + n^2] = (2n+1)(2n^2 + 2n + 1) \ ,$$

so that we can try either $2n + 1$ (which does not work) or $2n^2 + 2n + 1$ to find that $4n^4 + 1 = (2n^2 + 2n + 1)(2n^2 - 2n + 1)$ and

$$4(n+1)^4 + 1 = (2n^2 + 2n + 1)(2n^2 + 6n + 5) = (2n^2 + 2n + 1)[2(n+2)^2 - 2(n+2) + 1] \ .$$

125. Determine the set of complex numbers z which satisfy

$$\text{Im } (z^4) = (\text{Re } (z^2))^2 \ ,$$

and sketch this set in the complex plane. (*Note:* Im and Re refer respectively to the imaginary and real parts.)

Solution 1. Let $z = x + yi$ and $z^2 = u + vi$. Then $u = x^2 - y^2$, $v = 2xy$ and $z^4 = (u^2 - v^2) + 2uvi$. Im $(z^4) = (\text{Re } (z^2))^2$ implies that $2uv = u^2$. Thus, $u = 0$ or $u = 2v$. These reduce to $x^2 = y^2$ or $(x - 2y)^2 = 5y^2$, so that the locus consists of the points z on the lines determined by the equations $y = x$, $y = -x$, $y = (\sqrt{5} - 2)x$, $y = (-\sqrt{5} - 2)x$.

Solution 2. Let $z = r(\cos\theta + i\sin\theta)$; then $z^2 = r^2(\cos 2\theta + i\sin 2\theta)$ and $z^4 = r^4(\cos 4\theta + i\sin 4\theta)$. The condition is equivalent to

$$r^4 \sin 4\theta = (r^2 \cos 2\theta)^2 \Leftrightarrow 2\sin 2\theta \cos 2\theta = \cos^2 2\theta \ .$$

Hence $\cos 2\theta = 0$ or $\tan 2\theta = \frac{1}{2}$. The latter possibility leads to $\tan^2\theta + 4\tan\theta - 1 = 0$ or $\tan\theta = -2 \pm \sqrt{5}$. This yields the same result as in Solution 1.

Solution 3. Let $z = x + yi$. Then $z^2 = x^2 - y^2 + 2xyi$ and $z^4 = (x^4 - 6x^2y^2 + y^4) + 4xy(x^2 - y^2)i$. Then the condition in the problem is equivalent to

$$4xy(x^2 - y^2) = (x^2 - y^2)^2 \ ,$$

which in turn is equivalent to $y = \pm x$ or $y^2 + 4xy - x^2 = 0$, *i.e.*, $y = (-2 \pm \sqrt{5})x$.

126. Let n be a positive integer exceeding 1, and let n circles (*i.e.*, circumferences) of radius 1 be given in the plane such that no two of them are tangent and the subset of the plane formed by the union of them is connected. Prove that the number of points that belong to at least two of these circles is at least n.

Solution. Let Γ be the set of circles and S be the set of points belonging to at least two of them. For $C \in \Gamma$ and $s \in S \cap C$, define $f(s, C) = 1/k$, where k is the number of circles passing through s, including C. For $C \in \Gamma$ and $s \notin C$, define $f(s, C) = 0$. Observe that, for each $s \in S$,

$$\sum_{C \in \Gamma} f(s, C) = 1 \ .$$

Let $C \in \Gamma$; select $s \in S \cap C$ for which $f(s, C) = 1/k$ is minimum. Let $C = C_1, C_2, \cdots, C_k$ be the circles that contain s. These circles (apart from C) meet C in distinct points, so that

$$\sum_{s \in S} f(s, C) \geq \frac{1}{k} + \frac{k-1}{k} = 1 \ .$$

Hence the number of points in S is equal to

$$\sum_{s \in S} \sum_{C \in \Gamma} f(s, C) = \sum_{C \in \Gamma} \sum_{s \in S} f(s, C) \geq n \ .$$

Comment. The full force of the connectedness condition is not needed. It is required only that each circle intersect with at least one other circle.