Solutions.

178. Suppose that n is a positive integer and that $x_1, x_2, \cdots, x_n$ are positive real numbers such that $x_1 + x_2 + \cdots + x_n = n$. Prove that

$$\sum_{i=1}^{n} \sqrt[n]{ax_i + b} \leq a + b + n - 1$$

for every pair a, b of real numbers with each $ax_i + b$ nonnegative. Describe the situation when equality occurs.

Solution. Regarding $ax_i + b$ as a product with $n - 1$ ones, we use the arithmetic-geometric means inequality to obtain that

$$\sqrt[n]{ax_i + b} \leq \frac{(ax_i + b) + 1 + \cdots + 1}{n}$$

for $1 \leq i \leq n$, with equality if and only if $x_i = (1 - b)/a$. Adding these n inequalities yields the desired result.

179. Determine the units digit of the numbers a^2, b^2 and ab (in base 10 numeration), where

$$a = 2^{2002} + 3^{2002} + 4^{2002} + 5^{2002}$$

and

$$b = 3^1 + 3^2 + 3^3 + \cdots + 3^{2002} .$$

Solution. Observe that, for positive integer k, $2^{4k} \equiv 6$ and $3^{4k} \equiv 1$, modulo 10, so that $2^{2002} \equiv 6 \cdot 4 \equiv 4$, $3^{2002} \equiv 9$ and $4^{2002} \equiv 6$, modulo 10. Hence $a \equiv 4 + 9 + 6 + 5 \equiv 4$ and $a^2 \equiv 6$, modulo 10. Note that $b = (1/2)(3^{2003} - 3)$, and that $3^{2003} - 3 \equiv 7 - 3 = 4$, modulo 10. Since b is the sum of evenly many factors, it is even, and so $b \equiv 2$ and $b^2 \equiv 4$, modulo 10. Finally, $ab \equiv 4 \cdot 2 = 8$, modulo 10. Hence the units digits of a^2, b^2 and ab are respectively 6, 4 and 8.

180. Consider the function f that takes the set of complex numbers into itself defined by $f(z) = 3z + |z|$. Prove that f is a bijection and find its inverse.

Solution. Injection (one-one). Suppose that $z = x + yi$ and $w = u + vi$, and that $f(z) = f(w)$. Then

$$3x + 3yi + \sqrt{x^2 + y^2} = 3u + 3vi + \sqrt{u^2 + v^2} .$$

Equating imaginary parts yields that $y = v$, so that

$$3(x - u) = \sqrt{u^2 + y^2} - \sqrt{x^2 + y^2} = (u^2 - x^2)/(\sqrt{u^2 + y^2} + \sqrt{x^2 + y^2}) .$$

Suppose, if possible, that $u \neq x$. Then

$$3(\sqrt{x^2 + y^2} + x) = -[3(\sqrt{u^2 + y^2} + u] .$$

Since $\sqrt{x^2 + y^2} \geq |x|$, and $\sqrt{u^2 + y^2} \geq |u|$, we see that, unless $x = y = u = v = 0$, this equation is impossible as the left side is positive and the right is negative Thus, $x = u$.

Surjection (onto). Let $a + bi$ be an arbitrary complex number, and suppose that $f(x + yi) = a + bi$. It is straightforward to see that $f(z) = 0$ implies that $z = 0$, so we may assume that $a^2 + b^2 > 0$. We must have that

$$3x + \sqrt{x^2 + y^2} = a$$

and

$$3y = b .$$

Substituting $y = b/3$ into the first equation yields

$$\sqrt{9x^2 + b^2} = 3a - 9x \ .$$

For this equation to be solvable, it is necessary that $3x \leq a$. Squaring both sides of the equation leads to

$$72x^2 - 54ax + 9a^2 - b^2 = 0 \ .$$

When $x = a/3$ is substituted into the left side of the equation, we obtain $8a^2 - 18a^2 + 9a^2 - b^2 = -(a^2 + b^2) < 0$. This means that the two roots of the equation straddle $a/3$, so that exactly one of the roots satisfies the necessary condition $3x \leq a$. Hence, we must have

$$(x, y) = \left(\frac{9a - \sqrt{9a^2 + 8b^2}}{24}, \frac{b}{3} \right) \ .$$

Thus, the function is injective and surjective, and so it is a bijection.

181. Consider a regular polygon with n sides, each of length a, and an interior point located at distances a_1, a_2, $\cdots$, a_n from the sides. Prove that

$$a \sum_{i=1}^{n} \frac{1}{a_i} > 2\pi \ .$$

Solution. By constructing triangles from bases along the sides of the polygons to the point A, we see that the area of the polygon is equal to

$$\frac{aa_1}{2} + \frac{aa_2}{2} + \cdots + \frac{aa_n}{2} = \frac{a}{2} \sum_{i=1}^{n} a_i \ .$$

However, by constructing triangles whose bases are the sides of the polygons and whose apexes are at the centre of the polygon, we see that the area of the polygon is equal to $\frac{1}{4} na^2 \cot(\pi/n)$. Making use of the arithmetic-harmonic means inequality, we find that

$$\frac{a}{2} \cot \frac{\pi}{n} = \frac{1}{n} \sum_{i=1}^{n} a_i \geq \frac{n}{\sum_{i=1}^{n} 1/a_i} \ ,$$

from which

$$\sum_{i=1}^{n} \frac{1}{a_i} \geq \frac{2n \cdot \tan(\pi/n)}{a} \ .$$

Since $\tan x > x$ for $0 < x < \pi/2$, we have that $\tan(\pi/n) > (\pi/n)$, we obtain that

$$\sum_{i=1}^{n} \frac{1}{a_i} > \frac{2\pi}{a} \ .$$

182. Let M be an interior point of the equilateral triangle ABC with each side of unit length. Prove that

$$MA.MB + MB.MC + MC.MA \geq 1 \ .$$

Solution. Let the respective lengths of MA, MB and MC be x, y and z, and let the respective angles BMC, CMA and AMB be α, β and γ. Then $\alpha + \beta + \gamma = 2\pi$. Now

$$\begin{aligned}
\cos\alpha + \cos\beta + \cos\gamma &= 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} + 2\cos^2\frac{\gamma}{2} - 1 \\
&= -2\cos\frac{\gamma}{2}\cos\frac{\alpha-\beta}{2} + 2\cos^2\frac{\gamma}{2} - 1 \\
&= \frac{1}{2}\left[2\cos\frac{\gamma}{2} - \cos\frac{\alpha-\beta}{2}\right]^2 + \frac{1}{2}\sin^2\frac{\alpha-\beta}{2} - \frac{3}{2} \geq -\frac{3}{2} \ .
\end{aligned}$$

¿From the Law of Cosines applied to the triangles MBC, MCA and MAB, we convert this equation to

$$\frac{y^2+z^2-1}{2yz}+\frac{x^2+z^2-1}{2xz}+\frac{y^2+x^2-1}{2xy}\geq -\frac{3}{2}\ .$$

This simplifies to $(x+y+z)(xy+xz+yz)-(x+y+z)\geq 0$. Since $x+y+z\neq 0$, the result follows.

183. Simplify the expression

$$\frac{\sqrt{1+\sqrt{1-x^2}}((1+x)\sqrt{1+x}-(1-x)\sqrt{1-x})}{x(2+\sqrt{1-x^2})}\ ,$$

where $0<|x|<1$.

Solution. Observe that

$$\begin{aligned}\sqrt{1+\sqrt{1-x^2}} &= \sqrt{\frac{1+x+2\sqrt{1-x^2}+1-x}{2}}\\ &= \sqrt{\frac{(\sqrt{1+x}+\sqrt{1-x})^2}{2}}\\ &= \frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{2}}\ .\end{aligned}$$

Then, using the formula $a^3-b^3=(a-b)(a^2+ab+b^2)$, we find that the expression given in the problem is equal to

$$\begin{aligned}&\frac{(\sqrt{1+x}+\sqrt{1-x})(\sqrt{1+x}^3-\sqrt{1-x}^3)}{x\sqrt{2}(2+\sqrt{1-x^2})}\\ &=\frac{(\sqrt{1+x}+\sqrt{1-x})(\sqrt{1+x}-\sqrt{1-x})(1+x+\sqrt{1-x^2}+1-x)}{x\sqrt{2}(2+\sqrt{1-x^2})}\\ &=\frac{(1+x-1+x)(2+\sqrt{1-x^2})}{x\sqrt{2}(2+\sqrt{1-x^2})}\\ &=\frac{2x}{x\sqrt{2}}=\sqrt{2}\ .\end{aligned}$$

184. Using complex numbers, or otherwise, evaluate

$$\sin 10^\circ \sin 50^\circ \sin 70^\circ\ .$$

Solution. Let $z=\cos 20^\circ + i\sin 20^\circ$, so that $1/z=\cos 20^\circ - i\sin 20^\circ$. Then, by De Moivre's Theorem, $z^9=-1$. Now,

$$\sin 70^\circ = \cos 20^\circ = \frac{1}{2}(z+\frac{1}{z}) = \frac{z^2+1}{2z}\ ,$$

$$\sin 50^\circ = \cos 40^\circ = \frac{1}{2}(z^2+\frac{1}{z^2}) = \frac{z^4+1}{2z^2}\ ,$$

and

$$\sin 10^\circ = \cos 80^\circ = \frac{1}{2}(z^4+\frac{1}{z^4}) = \frac{z^8+1}{2z^4}\ .$$

Hence

$$\begin{aligned}\sin 10^\circ \sin 50^\circ \sin 70^\circ &= \frac{z^2+1}{2z}\cdot\frac{z^4+1}{2z^2}\cdot\frac{z^8+1}{2z^4}\\ &= \frac{1+z^2+z^4+z^6+z^8+z^{10}+z^{12}+z^{14}}{8z^7}\\ &= \frac{1-z^{16}}{8z^7(1-z^2)}\\ &= \frac{1-z^7z^9}{8(z^7-z^9)}\\ &= \frac{1+z^7}{8(z^7+1)} = \frac{1}{8}\ .\end{aligned}$$